Beyond Bird’s Nested Arrays IV

The Nested Hyper-Nested Array Notation, as evolved in the first three Beyond Bird’s Nested Arrays documents, was created for the ‘layman’ – it is made as easy to follow and understand as possible. What if I experimented by changing some of the rules, in particular, the ‘super-rule’ Angle Bracket Rule A5 on pages 24-25 of Beyond Bird’s Nested Arrays III? Is it possible for the limit ordinal of the Nested Hyper-Nested Arrays to be much greater than the Bachmann-Howard ordinal?

The separator with level $\varphi(\varepsilon_0, 0)$ was introduced on page 20 of Beyond Bird’s Nested Arrays I and originally written as [1 [1 \ 2]\ 2]. The rewriting of the [X]\ 1-hyperseparator (for an arbitrary array X) as [X¬2] at the beginning of Beyond Bird’s Nested Arrays II has led to the $\varphi(\varepsilon_0, 0)$ level separator being written as [1 [1 \ 2 ¬2] 2]. But the same ordinal can be achieved by putting a normal separator (or 0-hyperseparator) in place of the backslash (lowest 1-hyperseparator), as follows:

[1 [1 [1 \ 2] 2 ¬2] 2].

The [1 [1 \ 2 ¬2] 2] separator can represent a higher ordinal level, perhaps the highest level possible using the notation – this is achieved by going as far as possible using only normal separators in the ‘base layers’ of brackets ending in ‘¬2’ before introducing 1-hyperseparators (I have always pursued this policy in the ‘base layers’ of entire separators, though in layers containing n-hyperseparators where n ≥ 2, I have only furthered the development of n-hyperseparators whilst treating 1-hyperseparators much like normal separators). Rule A5 would need to be modified in order to accommodate this revision.

In order to signify the change in the rules and avoid confusion with the notation in the previous document, I have decided to use forward slashes instead of backslashes, so \ becomes / and $\backslash_n$ becomes $/_n$ (the n subscript can be omitted when n = 1). Subscripts are somewhat easier to read when placed after forward slashes and it is possible to ‘tuck’ subscripts underneath a forward slash when space is at a premium. The ¬ symbol (lowest 2-hyperseparator), which was generalised as $\backslash_2$ in Beyond Bird’s Nested Arrays III and now denotes $/_2$, is now written as the ~ symbol (or tilde); ~ is easier to read than $/_2$ in large separator expressions. Thus,

[1 [1 [1 \ 2] 2 ¬2] 2] now becomes [1 [1 [1 / 2] 2 ~2] 2],
[1 [1 \ 2 ¬2] 2] now becomes [1 [1 / 2 ~2] 2].

All the ordinal levels of the separators up to (but not including) [1 [1 [1 / 2] 2 ~2] 2] remain essentially the same as before. The ones listed on the first page of Beyond Bird’s Nested Arrays II (from [1 \ 2] onwards, now [1 / 2] onwards) are as follows:

[1 / 2] has level ε_0,
[1 [1 / 2] 2 / 2] has level $\varepsilon_0 2$,
[1 [1 [1 / 2] 2 / 2] 2 / 2] has level ε_0^ε_0,
[1 / 3] has level ε_1,
[1 / 4] has level ε_2,
[1 / 1, 2] has level ε_ω,
[1 / 1 [1 / 2] 2] has level $\varepsilon(\varepsilon_0)$,
[1 / 1 / 2] has level $\zeta_0 = \varphi(2, 0)$,
[1 / 1 / 1 / 2] has level $\varphi(3, 0)$,
[1 [2~2] 2] has level $\varphi(\omega, 0)$,
[1 [1, 2 ~2] 2] has level $\varphi(\omega$^$\omega, 0)$,
[1 [1 [2] 2 ~2] 2] has level $\varphi(\omega$^ω^$\omega, 0)$,
[1 [1 [1, 2] 2 ~2] 2] has level $\varphi(\omega$^ω^ω^$\omega, 0)$.

The ordinal levels of the revised separators from [1 [1 [1 / 2] 2 ~2] 2] onwards are as follows:

[1 [1 [1 / 2] 2 ~2] 2] has level $\varphi(\varepsilon_0, 0)$,
[1 [1 [1 / 2] 2 ~2] 3] has level $\varphi(\varepsilon_0, 1)$ (limit ordinal of $\varphi(\alpha, \varphi(\varepsilon_0, 0)+1)$ as $\alpha \to \varepsilon_0$),
[1 [1 [1 / 2] 2 ~2] 1, 2] has level $\varphi(\varepsilon_0, \omega)$,
[1 [1 [1 / 2] 2 ~2] 1 [1 / 2] 2] has level $\varphi(\varepsilon_0, \varepsilon_0)$,
[1 [1 [1 / 2] 2 ~2] 1 [1 [1 [1 / 2] 2 ~2] 2] 2] has level $\varphi(\varepsilon_0, \varphi(\varepsilon_0, 0))$,
[1 [1 [1 / 2] 2 ~2] 1 [1 [1 [1 / 2] 2 ~2] 1 [1 [1 [1 / 2] 2 ~2] 2] 2] 2] has level $\varphi(\varepsilon_0, \varphi(\varepsilon_0, \varphi(\varepsilon_0, 0)))$,
[1 [1 [1 / 2] 2 ~2] 1 / 2] has level $\varphi(\varepsilon_0+1, 0)$,
[1 [1 [1 / 2] 2 ~2] 1 [1 [1 / 2] 2 ~2] 2] has level $\varphi(\varepsilon_0 2, 0)$,
[1 [2 [1 / 2] 2 ~2] 2] has level $\varphi(\varepsilon_0\omega, 0)$,
[1 [1, 2 [1 / 2] 2 ~2] 2] has level $\varphi(\varepsilon_0\omega{}^\wedge\omega, 0)$,
[1 [1 [1 / 2] 3 ~2] 2] has level $\varphi(\varepsilon_0{}^\wedge 2, 0)$,
[1 [1 [1 / 2] 1 [1 / 2] 2 ~2] 2] has level $\varphi(\varepsilon_0{}^\wedge\varepsilon_0, 0)$,
[1 [1 [1 / 2] 1 [1 / 2] 1 [1 / 2] 2 ~2] 2] has level $\varphi(\varepsilon_0{}^\wedge\varepsilon_0{}^\wedge 2, 0)$,
[1 [1 [2 / 2] 2 ~2] 2] has level $\varphi(\varepsilon_0{}^\wedge\varepsilon_0{}^\wedge\omega, 0)$,
[1 [1 [1 [1 / 2] 2 / 2] 2 ~2] 2] has level $\varphi(\varepsilon_0{}^\wedge\varepsilon_0{}^\wedge\varepsilon_0, 0)$,
[1 [1 [1 / 3] 2 ~2] 2] has level $\varphi(\varepsilon_1, 0)$,
[1 [1 [1 / 1, 2] 2 ~2] 2] has level $\varphi(\varepsilon_\omega, 0)$,
[1 [1 [1 / 1 [1 / 2] 2] 2 ~2] 2] has level $\varphi(\varepsilon(\varepsilon_0), 0)$,
[1 [1 [1 / 1 / 2] 2 ~2] 2] has level $\varphi(\zeta_0, 0) = \varphi(\varphi(2, 0), 0)$,
[1 [1 [1 / 1 / 1 / 2] 2 ~2] 2] has level $\varphi(\varphi(3, 0), 0)$,
[1 [1 [1 [2~2] 2] 2 ~2] 2] has level $\varphi(\varphi(\omega, 0), 0)$,
[1 [1 [1 [1, 2 ~2] 2] 2 ~2] 2] has level $\varphi(\varphi(\omega{}^\wedge\omega, 0), 0)$,
[1 [1 [1 [1 [1 / 2] 2 ~2] 2] 2 ~2] 2] has level $\varphi(\varphi(\varepsilon_0, 0), 0)$,
[1 [1 [1 [1 [1 [1 [1 / 2] 2 ~2] 2] 2 ~2] 2] 2 ~2] 2] has level $\varphi(\varphi(\varphi(\varepsilon_0, 0), 0), 0)$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 / 2] 2 ~2] 2] 2 ~2] 2] 2 ~2] 2] 2 ~2] 2] has level $\varphi(\varphi(\varphi(\varphi(\varepsilon_0, 0), 0), 0), 0)$.

The sequence of separators starting with the last three has limit ordinal $\Gamma_0 = \theta(\Omega)$. The θ collapsing function (in single-argument form when the second argument is zero) is used to express the transfinite numbers beyond the Gamma ordinals ($\Gamma_\alpha = \theta(\Omega, \alpha)$).

I now have the [1 [1 / 2 ~2] 2] separator at level Γ_0. This entails a rather awkward double nested brackets procedure at a much lower level than previously, as I have avoided this procedure until level $\theta(\Omega{}^\wedge\Omega{}^\wedge\Omega)$. Hence, the revised separator expressions on Beyond Bird's Nested Arrays II would be more unwieldy, particularly where large expressions are repeatedly 'plugged' into normal separators (somewhere within the entire separators), in order to determine limit ordinals.

Rule A5 (pages 24-25 of Beyond Bird's Nested Arrays III), has five subrules (a-e) and two counters – a branching counter (i) and a nesting counter (j). Two of the five subrules involve branching or nesting up a level and repeating the five subrules for the next level – subrule c increments i by 1 and resets j to 1, while subrule d increments j by 1 – all other subrules terminate.

I now make some changes to Rule A5. In Rules A5c and A5d, each of $[A_{i+1,1,i*}]$ is a normal separator and each of $[A_{i,j+1,i*}]$ is a 1- or higher order hyperseparator. This is due to the fact that the revised 1-hyperseparators are now proper hyperseparators, truly distinguishable from normal separators and acting much more like the 2-hyperseparators in the previous notation. The forward slash (/) is now very much different from the [1 / 2] separator.

In fact, Rule A5c can now be abolished and removed from the rules (with the default subrule, Rule A5e, applying in these cases). This is because it is really designed for the cases when $[A_{i+1,1,p_{i+1,1}}]$ is a

1-hyperseparator under the previous notation, and we can exit Rule A5 with A5e and start afresh by considering all of the Angle Bracket Rules when $[A_{i+1,1,p_{i+1,1}}]$ is a normal separator, as the angle bracket string ‘b $\langle A_{i+1,1,p_{i+1,1}}' \rangle$ b’ (where $A_{i+1,1,p_{i+1,1}}'$ is identical to $A_{i+1,1,p_{i+1,1}}$ except that the first entry has been reduced by 1) does not contain any 2- or higher order hyperseparators – only Rules A1, A5 or A6 can apply when evaluating ‘b $\langle A_{i+1,1,p_{i+1,1}}' \rangle$ b’, and when Rule A5 applies ($A_{i+1,1,p_{i+1,1}}'$ begins with 0 and at least one other entry), the #* string is blank. This means that the branching counter (i) is no longer necessary and the nesting counter (j) is now denoted by the variable i.

Rule A5a now comes into operation for all values of i (previously j); it becomes a multinesting subrule (like A5b, which will be changed later). As I hope that Rule A5b will be absorbed into A5a, Rule A5b is now known as Rule A5a*, while Rules A5d and A5e are now renamed Rules A5b and A5c respectively. Since the revised Rule A5b works equally well when $[A_{i,p_i}]$ is a 1- or higher order hyperseparator, rather than $[A_{i+1,p_{i+1}}]$ (previously $[A_{i,j+1,p_{i,j+1}}]$), I have decided to make another minor alteration to that rule.

Angle Bracket Rule A5 (excluding A5a*) now reads as follows:-

Rule A5 (Rules A1-4 do not apply, first entry is 0, separator immediately prior to next non-1 entry (c_1) is $[A_{1,p_1}]$):

$$\text{‘}a \langle 0\ [A_{1,1}]\ 1\ [A_{1,2}] \ldots 1\ [A_{1,p_1}]\ c_1\ \#_1\ \#^* \rangle\ b\text{’} = \text{‘}a \langle S_1\ \#^* \rangle\ b\text{’},$$

where $p_1 \geq 1$, each of $[A_{1,j}]$ is either a normal separator or 1-hyperseparator, $\#_1$ contains no 2- or higher order hyperseparators in its base layer and #* is either an empty string or begins with a 2- or higher order hyperseparator.
Set i = 1 and follow Rules A5a-c (a, a* and c are terminal, b is not).

Rule A5a (separator $[A_{i,p_i}] = [1\ /_2\ 2] = /$):

$$S_i = \text{‘}R_{b,i}\text{’}.$$

For $n > 1$ and $1 \leq k < i$,

$$R_{n,i} = \text{‘}b \langle A_{i,1}' \rangle\ b\ [A_{i,1}]\ b \langle A_{i,2}' \rangle\ b\ [A_{i,2}] \ldots b \langle A_{i,p_i-1}' \rangle\ b\ [A_{i,p_i-1}]\ b \langle R_{n-1,1} \rangle\ b\ /\ c_i\text{-}1\ \#_i\text{’},$$

$$R_{n,k} = \text{‘}b \langle A_{k,1}' \rangle\ b\ [A_{k,1}]\ b \langle A_{k,2}' \rangle\ b\ [A_{k,2}] \ldots b \langle A_{k,p_k-1}' \rangle\ b\ [A_{k,p_k-1}]\ b \langle R_{n,k+1} \rangle\ b\ [A_{k,p_k}]\ c_k\text{-}1\ \#_k\text{’},$$

$$R_{1,1} = \text{‘}0\text{’}.$$

Rule A5b (Rules A5a-a* do not apply, separator $[A_{i,p_i}] = [1\ [A_{i+1,1}]\ 1\ [A_{i+1,2}] \ldots 1\ [A_{i+1,p_{i+1}}]\ c_{i+1}\ \#_{i+1}]$, which is a 1- or higher order hyperseparator, where $p_{i+1} \geq 1$ and $c_{i+1} \geq 2$):

$$S_i = \text{‘}b \langle A_{i,1}' \rangle\ b\ [A_{i,1}]\ b \langle A_{i,2}' \rangle\ b\ [A_{i,2}] \ldots b \langle A_{i,p_i-1}' \rangle\ b\ [A_{i,p_i-1}]\ b \langle S_{i+1} \rangle\ b\ [A_{i,p_i}]\ c_i\text{-}1\ \#_i\text{’}.$$

Increment i by 1 and repeat Rules A5a-c.

Rule A5c (Rules A5a-b do not apply):

$$S_i = \text{‘}b \langle A_{i,1}' \rangle\ b\ [A_{i,1}]\ b \langle A_{i,2}' \rangle\ b\ [A_{i,2}] \ldots b \langle A_{i,p_i}' \rangle\ b\ [A_{i,p_i}]\ c_i\text{-}1\ \#_i\text{’}.$$

(When i = 1, Rule A5a remains the same as before, otherwise, it involves i layers of angle brackets per iteration of n in the $R_{n,k}$ function. In Rule A5b, $[A_{i,p_i}]$ being a 1- or higher order hyperseparator implies that each of $[A_{i,j}]$ is a 1- or higher hyperseparator.)

I can now redefine the Γ_0 level separator:

$$\{a, b\ [1\ [1\ /\ 2\ \sim 2]\ 2]\ 2\} = \{a \langle b \langle b \langle b \langle b \langle \ldots \langle b \langle b \langle b \langle b \sim 2 \rangle\ b \rangle\ b \sim 2 \rangle\ b \rangle \ldots \rangle\ b \sim 2 \rangle\ b \rangle\ b \sim 2 \rangle\ b \rangle\ b\}$$

(with 2b-2 pairs of angle brackets and b-1 ~’s).

The above is verified using the revised Rule A5:

{a, b [1 [1 / 2 ~2] 2] 2} = {a ‹0 [1 / 2 ~2] 2› b}
= {a ‹S_1› b}.

Since $p_1 = 1$, $c_1 = 2$, $\#_1 = \#^* =$ ‘’ (blank),
[$A_{1,1}$] = [1 / 2 ~2] (1-hyperseparator),
$p_2 = 1$, $c_2 = 2$, $\#_2$ = ‘~2’,
[$A_{2,1}$] = / (1-hyperseparator),

by Rule A5b,
S_1 = ‘b ‹S_2› b’,

and by Rule A5a,
S_2 = ‘$R_{b,2}$’,
$R_{n,2}$ = ‘b ‹$R_{n-1,1}$› b ~2’,
$R_{n,1}$ = ‘b ‹$R_{n,2}$› b’ (so, $S_1 = R_{b,1}$)
= ‘b ‹b ‹$R_{n-1,1}$› b ~2› b’,
$R_{1,1}$ = ‘0’.

It follows that,
{a, b [1 [1 / 2 ~2] 2] 2} = {a ‹R_b› b},
where R_n = ‘b ‹b ‹R_{n-1}› b ~2› b’,
R_1 = ‘0’.

The ordinal levels of the revised separators from [1 [1 / 2 ~2] 2] onwards are as follows:

[1 [1 / 2 ~2] 2] has level $\Gamma_0 = \theta(\Omega)$,
[1 [1 / 2 ~2] 3] has level $\Gamma_1 = \theta(\Omega, 1)$,
[1 [1 / 2 ~2] 1 / 2] has level θ(Ω+1) (first fixed point of $\alpha = \Gamma_\alpha$),
[1 [1 / 2 ~2] 1 [1 / 2 ~2] 2] has level θ(Ω2),
[1 [2 / 2 ~2] 2] has level θ(Ωω),
[1 [1 [1 / 2] 2 / 2 ~2] 2] has level $\theta(\Omega\varepsilon_0)$,
[1 [1 [1 [1 / 2 ~2] 2] 2 / 2 ~2] 2] has level $\theta(\Omega\Gamma_0)$,
[1 [1 [1 [1 [1 [1 / 2 ~2] 2] 2 / 2 ~2] 2] 2 / 2 ~2] 2] has level $\theta(\Omega\theta(\Omega\Gamma_0))$,
[1 [1 / 3 ~2] 2] has level θ(Ω^2),
[1 [1 [1 / 2] 2 / 3 ~2] 2] has level $\theta((\Omega\text{^}2)\varepsilon_0)$,
[1 [1 [1 [1 / 3 ~2] 2] 2 / 3 ~2] 2] has level θ((Ω^2)θ(Ω^2)),
[1 [1 / 4 ~2] 2] has level θ(Ω^3),
[1 [1 / 1, 2 ~2] 2] has level θ(Ω^ω) (small Veblen ordinal),
[1 [1 / 1 [1 / 2] 2 ~2] 2] has level $\theta(\Omega\text{^}\varepsilon_0)$,
[1 [1 / 1 [1 [1 / 2 ~2] 2] 2 ~2] 2] has level $\theta(\Omega\text{^}\Gamma_0)$,
[1 [1 / 1 [1 [1 / 1 [1 [1 / 2 ~2] 2] 2 ~2] 2] 2 ~2] 2] has level $\theta(\Omega\text{^}\theta(\Omega\text{^}\Gamma_0))$,
[1 [1 / 1 [1 [1 / 1 [1 [1 / 1 [1 [1 / 2 ~2] 2] 2 ~2] 2] 2 ~2] 2] 2 ~2] 2] has level $\theta(\Omega\text{^}\theta(\Omega\text{^}\theta(\Omega\text{^}\Gamma_0)))$.

[1 [1 / 1 / 2 ~2] 2] has level θ(Ω^Ω) (large Veblen ordinal),
[1 [1 / 2 / 2 ~2] 2] has level θ(Ω^(Ω+1)),
[1 [1 / 1 [1 / 2] 2 / 2 ~2] 2] has level $\theta(\Omega\text{^}(\Omega+\varepsilon_0))$,
[1 [1 / 1 [1 [1 / 1 / 2 ~2] 2] 2 / 2 ~2] 2] has level θ(Ω^(Ω+θ(Ω^Ω))),
[1 [1 / 1 / 3 ~2] 2] has level θ(Ω^(Ω2)),
[1 [1 / 1 / 1 [1 / 2] 2 ~2] 2] has level $\theta(\Omega\text{^}(\Omega\varepsilon_0))$,
[1 [1 / 1 / 1 [1 [1 / 1 / 2 ~2] 2] 2 ~2] 2] has level θ(Ω^(Ωθ(Ω^Ω))),
[1 [1 / 1 / 1 / 2 ~2] 2] has level θ(Ω^Ω^2),
[1 [1 [2~2] 2 ~2] 2] has level θ(Ω^Ω^ω),
[1 [1 [1 [1 / 2] 2 ~2] 2 ~2] 2] has level $\theta(\Omega\text{^}\Omega\text{^}\varepsilon_0)$,

[1 [1 [1 [1 [1 / 1 / 2 ~2] 2] 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^θ(Ω^Ω)),
[1 [1 [1 [1 [1 [1 [1 [1 / 1 / 2 ~2] 2] 2 ~2] 2 ~2] 2] 2 ~2] 2 ~2] 2] has level
θ(Ω^Ω^θ(Ω^Ω^θ(Ω^Ω))).

[1 [1 [1 / 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω),
[1 [1 [1 / 2 ~2] 3 ~2] 2] has level θ(Ω^((Ω^Ω)2)),
[1 [1 [1 / 2 ~2] 1 / 2 ~2] 2] has level θ(Ω^Ω^(Ω+1)),
[1 [1 [1 / 2 ~2] 1 [1 / 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^(Ω2)),
[1 [1 [2 / 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^(Ωω)),
[1 [1 [1 [1 / 2] 2 / 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^(Ωε_0)),
[1 [1 [1 [1 [1 [1 / 2 ~2] 2 ~2] 2] 2 / 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^(Ωθ(Ω^Ω^Ω))),
[1 [1 [1 / 3 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω^2),
[1 [1 [1 / 1 [1 / 2] 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω^ε_0),
[1 [1 [1 / 1 [1 [1 [1 / 2 ~2] 2 ~2] 2] 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω^θ(Ω^Ω^Ω)).

[1 [1 [1 / 1 / 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω^Ω),
[1 [1 [1 / 1 / 3 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω^(Ω2)),
[1 [1 [1 / 1 / 1 / 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω^Ω^2),
[1 [1 [1 [2~2] 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω^Ω^ω),
[1 [1 [1 [1 [1 / 2] 2 ~2] 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω^Ω^ε_0),
[1 [1 [1 [1 [1 [1 [1 / 1 / 2 ~2] 2 ~2] 2] 2 ~2] 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω^Ω^θ(Ω^Ω^Ω^Ω)).

A pattern can be seen:
[1 [1 / 2 ~2] 2] has level Γ_0 = θ(Ω),
[1 [1 / 1 / 2 ~2] 2] has level θ(Ω^Ω),
[1 [1 [1 / 2 ~2] 2 ~2] 2] has level θ(Ω^Ω^Ω),
[1 [1 [1 / 1 / 2 ~2] 2 ~2] 2] has level θ(Ω^^4) = θ(Ω^Ω^Ω^Ω),
[1 [1 [1 [1 / 2 ~2] 2 ~2] 2 ~2] 2] has level θ(Ω^^5),
[1 [1 [1 [1 / 1 / 2 ~2] 2 ~2] 2 ~2] 2] has level θ(Ω^^6),
[1 [1 [1 [1 [1 / 2 ~2] 2 ~2] 2 ~2] 2 ~2] 2] has level θ(Ω^^7).
The sequence of separators has limit ordinal θ($\varepsilon_{\Omega+1}$) (Bachmann-Howard ordinal), which is achieved without the need for any subscripts in the slashes, although the ~ symbol is the $/_2$ symbol, the lowest 2-hyperseparator. (I have used ~ for $/_2$ as ~ is easier to read in large separator expressions.)

In general, with n layers of square brackets (n ≥ 2),
[1 [1 [1 [... [1 [1 / 2 ~2] 2 ~2] ...] 2 ~2] 2 ~2] 2] has level θ(Ω^^(2n-3)),
[1 [1 [1 [... [1 [1 / 1 / 2 ~2] 2 ~2] ...] 2 ~2] 2 ~2] 2] has level θ(Ω^^(2n-2)).

The θ(Ωω) level separator
{a, b [1 [2 / 2 ~2] 2] 2} = {a ‹0 [2 / 2 ~2] 2› b}
= {a ‹S_1› b}.
Since p_1 = 1, c_1 = 2, $\#_1$ = #* = ‘’ (blank),
$[A_{1,1}]$ = [2 / 2 ~2] (1-hyperseparator),
by Rule A5c,
S_1 = ‘b ‹1 / 2 ~2› b’,
and by Rule A6,
S_1 = ‘b ‹0 / 2 ~2› b [1 / 2 ~2] b ‹0 / 2 ~2› b [1 / 2 ~2] ... [1 / 2 ~2] b ‹0 / 2 ~2› b’
(with b ‘b ‹0 / 2 ~2› b’ strings).
By Rule A5 (again),

‘b ‹0 / 2 ~2› b’ = ‘b ‹S_1 ~2› b’.

Since $p_1 = 1$, $c_1 = 2$, $\#_1$ = ‘’ (blank), #* = ‘~2’,

$[A_{1,1}]$ = / (1-hyperseparator),

by Rule A5a,

S_1 = ‘$R_{b,1}$’,

$R_{n,1}$ = ‘b ‹$R_{n-1,1}$› b’,

$R_{1,1}$ = ‘0’.

It follows that,

{a, b [1 [2 / 2 ~2] 2] 2} = {a ‹ S [1 / 2 ~2] S [1 / 2 ~2] ... [1 / 2 ~2] S › b}

(with b S strings),

where S = ‘b ‹0 / 2 ~2› b’

= ‘b ‹b ‹b ‹b ‹ ... ‹b ‹b› b› ... › b› b› b ~2› b’

(with b-1 pairs of angle brackets and one ~ symbol).

The main reason for excluding 2- or higher order hyperseparators from the $\#_1$ string (placing them in #* instead) is that we would have the second-layer Γ_0 level

‘b ‹0 / 2 ~2› b’ = ‘b ‹b ‹b ‹ ... ‹b ‹b~2› b~2› ... › b~2› b~2› b’

(with b-1 pairs of angle brackets and b-1 ~’s),

which produces an angle bracket array associated with a much higher ordinal (Bachmann-Howard ordinal, see later). Since ‘b ‹0 / 2 ~2› b’ is essentially second-layer or above, it is acceptable to have it yielding an angle bracket array associated with a much lower ordinal, such as $\varphi(\varepsilon_0, 0)$, but which is large enough to form a basis for a sequence of ordinals towards the desired one (e.g. Γ_0 is the limit of the sequence $\varphi(\varepsilon_0, 0)$, $\varphi(\varphi(\varepsilon_0, 0), 0)$, $\varphi(\varphi(\varphi(\varepsilon_0, 0), 0), 0)$ etc.) – this should result in no changes to the ordinal levels of the separators.

The $\theta(\Omega\text{^}\varepsilon_0)$ level separator

{a, b [1 [1 / 1 [1 / 2] 2 ~2] 2] 2} = {a ‹0 [1 / 1 [1 / 2] 2 ~2] 2› b}

= {a ‹S_1› b}.

Since $p_1 = 1$, $c_1 = 2$, $\#_1$ = #* = ‘’ (blank),

$[A_{1,1}]$ = [1 / 1 [1 / 2] 2 ~2] (1-hyperseparator),

$p_2 = 2$, $c_2 = 2$, $\#_2$ = ‘~2’,

$[A_{2,1}]$ = / (1-hyperseparator),

$[A_{2,2}]$ = [1 / 2] (0-hyperseparator),

by Rule A5b,

S_1 = ‘b ‹S_2› b’,

and by Rule A5c,

S_2 = ‘b / b ‹0 / 2› b ~2’ (since ‘b ‹$A_{2,1}$’› b’ = ‘b ‹0~2› b’ = ‘b’).

Putting this together, we obtain,

{a, b [1 [1 / 1 [1 / 2] 2 ~2] 2] 2} = {a ‹b ‹b / b ‹0 / 2› b ~2› b› b}.

After exiting the entire Rule A5, we then re-enter it in order to evaluate

‘b ‹0 / 2› b’ = ‘b ‹S_1› b’.

Since $p_1 = 1$, $c_1 = 2$, $\#_1$ = #* = ‘’ (blank),

$[A_{1,1}]$ = / (1-hyperseparator),

by Rule A5a,

S_1 = ‘$R_{b,1}$’,

$R_{n,1}$ = ‘b ‹$R_{n-1,1}$› b’,

$R_{1,1}$ = ‘0’.

It follows that,

{a, b [1 [1 / 1 [1 / 2] 2 ~2] 2] 2} = {a ‹b ‹b / S ~2› b› b},

where S = ‘b ‹b ‹b ‹ ... ‹b ‹b› b› ... › b› b› b’ (with b-1 pairs of angle brackets).
The previous Rule A5c (Beyond Bird’s Nested Arrays III) is redundant as similar results are obtained for general [$A_{1,1}$] containing one or more 1’s and hyperseparators followed by one or more 1’s and normal separators.

The θ(Ω^Ω^Ω) level separator

{a, b [1 [1 [1 / 2 ~2] 2 ~2] 2] 2} = {a ‹0 [1 [1 / 2 ~2] 2 ~2] 2› b}
= {a ‹S_1› b}.

Since p_1 = 1, c_1 = 2, $\#_1$ = #* = ‘’ (blank),
[$A_{1,1}$] = [1 [1 / 2 ~2] 2 ~2] (1-hyperseparator),
p_2 = 1, c_2 = 2, $\#_2$ = ‘~2’,
[$A_{2,1}$] = [1 / 2 ~2] (1-hyperseparator),
p_3 = 1, c_3 = 2, $\#_3$ = ‘~2’,
[$A_{3,1}$] = / (1-hyperseparator),

by Rule A5b (twice – with i = 1 and i = 2),
S_1 = ‘b ‹S_2› b’
= ‘b ‹b ‹S_3› b ~2› b’,

and by Rule A5a (with i = 3),
S_3 = ‘$R_{b,3}$’,
$R_{n,3}$ = ‘b ‹$R_{n-1,1}$› b ~2’,
$R_{n,1}$ = ‘b ‹$R_{n,2}$› b’
= ‘b ‹b ‹$R_{n,3}$› b ~2› b’
= ‘b ‹b ‹b ‹$R_{n-1,1}$› b ~2› b ~2› b’,
$R_{1,1}$ = ‘0’.

As S_1 = $R_{b,1}$, it follows that,
{a, b [1 [1 [1 / 2 ~2] 2 ~2] 2] 2} = {a ‹R_b› b},
where R_n = ‘b ‹b ‹b ‹R_{n-1}› b ~2› b ~2› b’,
R_1 = ‘0’.

Rule A5a involves three layers of angle brackets per iteration of n in the $R_{n,k}$ function, with two of these containing ‘~2’ at the end of the angle brackets.

With k layers of square brackets (k ≥ 2), the θ(Ω^^(2k-3)) level separator
{a, b [1 [1 [1 [... [1 [1 / 2 ~2] 2 ~2] ...] 2 ~2] 2 ~2] 2] 2} = {a ‹R_b› b},
where R_n = ‘b ‹b ‹b ‹ ... ‹b ‹b ‹R_{n-1}› b~2› b~2› ... › b~2› b~2› b’
(with k pairs of angle brackets and k-1 ~’s),
R_1 = ‘0’.

Rule A5a involves k layers of angle brackets per iteration of n in the $R_{n,k}$ function, with all but one of these containing ‘~2’ at the end of the angle brackets.

We can proceed beyond the Bachmann-Howard ordinal by introducing the [1 [1~3] 2] separator (previously at the Γ_0 level). In terms of the array notation, its definition remains the same as before:
{a, b [1 [1~3] 2] 2} = {a ‹0 [1~3] 2› b}
= {a ‹b ‹b ‹b ‹ ... ‹b ‹b~2› b~2› ... › b~2› b~2› b› b}
(with b pairs of angle brackets and b-1 ~’s).

This is equivalent to
{a, b [1 [1 [1 [... [1 [1, 2 ~2] 2 ~2] ...] 2 ~2] 2 ~2] 2] 2} (with b pairs of square brackets),
where the separator has level θ(Ω^Ω^Ω^...^Ω^ω^ω) (with 2b-4 Ω’s).

Unlike with the notation in the previous document, I intend to allow the possibility of mixing hyperseparators of different levels on the same 'nested level' of a separator, for example,

[# [a $/_3$ b $/_5$ c $/_2$ d] #*] (# and #* represent the remainder of the entire separator),

as this system leads to even more gigantic numbers. 1-hyperseparators are created and developed by repeatedly nesting normal separators (either the entire separator array or a normal separator within it); 2-hyperseparators are obtained and furthered by repeatedly nesting 1-hyperseparators; 3-hyperseparators are launched and advanced by repeatedly nesting 2-hyperseparators; and so on. In general, n-hyperseparators are introduced and grown by repeatedly nesting (n-1)-hyperseparators; there is a proper hierarchy of hyperseparator levels, where an n-hyperseparator outranks any (n-1)-hyperseparator. With a 'nested layer' ending in, say, '$/_m$ k', the aim is to 'build up' the hyperseparators preceding it (from normal, 1-hyperseparator, 2-hyperseparator etc.) until one achieves an (m-1)-hyperseparator immediately preceding the '$/_m$ k' that is an exact copy of that 'nested layer' (up to a certain number of levels), meaning that the (m-1)-hyperseparator would also end in '$/_m$ k' and preceded by a similar (m-1)-hyperseparator, and so on, to a given number of levels. The limit of this would be represented by the original 'nested layer' being [1 $/_m$ k+1].

The 'super-rule' Angle Bracket Rule A5 effectively has three subrules, according to the composition of the $[A_{i,p_i}]$ separator:

A5a: $[A_{i,p_i}] = [1\ /_{m+1}\ 2] = /_m$ (m ≥ 1).

A5b: $[A_{i,p_i}] = [1\ [A_{i+1,1}]\ 1\ [A_{i+1,2}]\ \ldots\ 1\ [A_{i+1,p_{i+1}}]\ c_{i+1}\ \#_{i+1}]$,
which is an m-hyperseparator ($p_{i+1} \geq 1$, $c_{i+1} \geq 2$, m ≥ 1).

A5c: All other scenarios of $[A_{i,p_i}]$.

The Rule now uses a set of x tally counters (or t-counters), where x is the highest hyperseparator level found within the $[A_{1,p_1}]$ separator; this highest level is the $/_x$ symbol. The t-counters are labelled from t_1 to t_x, with t_k representing the number of consecutive applications of Rule A5b where $[A_{i,p_i}]$ is a k- or higher order hyperseparator (t_k is a (k+)-hyperseparator counter). It begins with i set to 1 and all of $t_1, t_2, \ldots, t_x$ set to 0. When Rule A5b needs to be executed and $[A_{i,p_i}]$ is an m-hyperseparator, all of i, $t_1, t_2, \ldots, t_m$ are incremented by 1, while all of $t_{m+1}, t_{m+2}, \ldots, t_x$ are reset to 0 and Rules A5a-c are repeated for $[A_{i,p_i}]$ after $i = t_1+1$ is increased by 1. The Rule A5 procedure is terminated when either of the other two subrules apply. When $[A_{i,p_i}]$ is the m-hyperseparator forward slash symbol, Rule A5a (now absorbing Rule A5a*) is performed for the highest t_m+1 layers of the angle bracket array in the initial part of A5 (from layers $s = i-t_m$ to i; all layers when m = 1); this involves repeatedly nesting these t_m+1 layers of angle brackets b-1 times, or t_m+1 layers per iteration of n in the R_n function (n decrements by one b-1 times, as R_1 = '0').

The revised Rule A5a reads as follows:-

Rule A5a (separator $[A_{i,p_i}] = [1\ /_{m+1}\ 2] = /_m$, where m ≥ 1):

$s = i-t_m$,

S_i = '$R_{b,i}$'.

For n > 1 and s ≤ k < i,

$R_{n,i}$ = 'b ‹$A_{i,1}$'› b [$A_{i,1}$] b ‹$A_{i,2}$'› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}'› b [A_{i,p_i-1}] b ‹$R_{n-1,s}$› b $/_m$ c_i-1 $\#_i$',

$R_{n,k}$ = 'b ‹$A_{k,1}$'› b [$A_{k,1}$] b ‹$A_{k,2}$'› b [$A_{k,2}$] ... b ‹A_{k,p_k-1}'› b [A_{k,p_k-1}] b ‹$R_{n,k+1}$› b [A_{k,p_k}] c_k-1 $\#_k$',

$R_{1,s}$ = '0'.

The initial part of Rule A5 ends with the line:-

Set i to 1 and $t_1, t_2, \ldots, t_x$ to 0 (where x is the highest subscript to a forward slash within $[A_{1,p_1}]$), and follow Rules A5a-c (a and c are terminal, b is not). (Note that $i = t_1+1$ throughout.)

In Rule A5b, $[A_{i,p_i}]$ is an m-hyperseparator, where m ≥ 1. This rule ends with the line:-
Increment i, $t_1, t_2, \ldots, t_m$ by 1; reset $t_{m+1}, t_{m+2}, \ldots, t_x$ to 0 and repeat Rules A5a-c. ($i = t_1+1$.)

The Bachmann-Howard level separator can now be verified using the revised Rule A5:

{a, b [1 [1~3] 2] 2} = {a ‹0 [1~3] 2› b}
= {a ‹S_1› b}.

Since $p_1 = 1$, $c_1 = 2$, $\#_1 = \#^* =$ ‘’ (blank),
$[A_{1,1}]$ = [1~3] (1-hyperseparator),
$p_2 = 1$, $c_2 = 3$, $\#_2 =$ ‘’ (blank),
$[A_{2,1}]$ = ~ (= $/_2$) (2-hyperseparator),

by Rule A5b (m = 1),
S_1 = ‘b ‹S_2› b’,
$t_1 = 1$, $t_2 = 0$,

and by Rule A5a (m = 2, s = 2),
S_2 = ‘$R_{b,2}$’,
$R_{n,2}$ = ‘b ‹$R_{n-1,2}$› b ~2’,
$R_{1,2}$ = ‘0’.

It follows that,
{a, b [1 [1~3] 2] 2} = {a ‹b ‹R_b› b› b},
where R_n = ‘b ‹R_{n-1}› b ~2’,
R_1 = ‘0’.

The ordinal levels of the revised separators from [1 [1~3] 2] onwards are as follows:

[1 [1~3] 2] has level $\theta(\varepsilon_{\Omega+1})$,
[2 [1~3] 2] has level $\theta(\varepsilon_{\Omega+1})+1$,
[1 [1 / 2] 2 [1~3] 2] has level $\theta(\varepsilon_{\Omega+1})+\varepsilon_0$,
[1 [1 [1~3] 2] 2 [1~3] 2] has level $\theta(\varepsilon_{\Omega+1})2$,
[1 [1 [1 [1~3] 2] 2 [1~3] 2] 2 [1~3] 2] has level $\theta(\varepsilon_{\Omega+1})$^$\theta(\varepsilon_{\Omega+1})$,
[1 / 2 [1~3] 2] has level $\varepsilon(\theta(\varepsilon_{\Omega+1})+1) = \theta(1, \theta(\varepsilon_{\Omega+1})+1)$,
[1 [1 / 2 ~2] 2 [1~3] 2] has level $\Gamma(\theta(\varepsilon_{\Omega+1})+1) = \theta(\Omega, \theta(\varepsilon_{\Omega+1})+1)$,
[1 [1 / 1 / 2 ~2] 2 [1~3] 2] has level $\theta(\Omega$^$\Omega, \theta(\varepsilon_{\Omega+1})+1)$,
[1 [1 [1 / 2 ~2] 2 ~2] 2 [1~3] 2] has level $\theta(\Omega$^Ω^$\Omega, \theta(\varepsilon_{\Omega+1})+1)$,
[1 [1~3] 3] has level $\theta(\varepsilon_{\Omega+1}, 1)$
(limit ordinal of $\theta(\alpha, \theta(\varepsilon_{\Omega+1})+1) = \theta(\alpha, \theta(\theta(\varepsilon_{\Omega+1}), 0)+1)$ as $\alpha \to \varepsilon_{\Omega+1}$),
[1 [1~3] 4] has level $\theta(\varepsilon_{\Omega+1}, 2)$,
[1 [1~3] 1 [1 / 2] 2] has level $\theta(\varepsilon_{\Omega+1}, \varepsilon_0)$,
[1 [1~3] 1 [1 [1~3] 2] 2] has level $\theta(\varepsilon_{\Omega+1}, \theta(\varepsilon_{\Omega+1}))$,
[1 [1~3] 1 [1 [1~3] 1 [1 [1~3] 2] 2] 2] has level $\theta(\varepsilon_{\Omega+1}, \theta(\varepsilon_{\Omega+1}, \theta(\varepsilon_{\Omega+1})))$.

[1 [1~3] 1 / 2] has level $\theta(\varepsilon_{\Omega+1}+1)$,
[1 [1~3] 1 [2~2] 2] has level $\theta(\varepsilon_{\Omega+1}+\omega)$,
[1 [1~3] 1 [1 / 2 ~2] 2] has level $\theta(\varepsilon_{\Omega+1}+\Omega)$,
[1 [1~3] 1 [1 / 1 / 2 ~2] 2] has level $\theta(\varepsilon_{\Omega+1}+\Omega$^$\Omega)$,
[1 [1~3] 1 [1 [1 / 2 ~2] 2 ~2] 2] has level $\theta(\varepsilon_{\Omega+1}+\Omega$^$\Omega$^$\Omega)$,
[1 [1~3] 1 [1~3] 2] has level $\theta(\varepsilon_{\Omega+1}2)$,
[1 [1~3] 1 [1~3] 1 [1~3] 2] has level $\theta(\varepsilon_{\Omega+1}3)$,
[1 [2~3] 2] has level $\theta(\varepsilon_{\Omega+1}\omega)$,
[1 [1 [1 [1~3] 2] 2 ~3] 2] has level $\theta(\varepsilon_{\Omega+1}\theta(\varepsilon_{\Omega+1}))$,
[1 [1 [1 [1 [1 [1~3] 2] 2 ~3] 2] 2 ~3] 2] has level $\theta(\varepsilon_{\Omega+1}\theta(\varepsilon_{\Omega+1}\theta(\varepsilon_{\Omega+1})))$,

[1 [1 / 2 ~3] 2] has level θ(ε$_{\Omega+1}$Ω),
[1 [1 / 1 / 2 ~3] 2] has level θ(ε$_{\Omega+1}$(Ω^Ω)),
[1 [1 [1 / 2 ~2] 2 ~3] 2] has level θ(ε$_{\Omega+1}$(Ω^Ω^Ω)),
[1 [1 [1 / 1 / 2 ~2] 2 ~3] 2] has level θ(ε$_{\Omega+1}$(Ω^Ω^Ω^Ω)),
[1 [1 [1 [1 / 2 ~2] 2 ~2] 2 ~3] 2] has level θ(ε$_{\Omega+1}$(Ω^Ω^Ω^Ω^Ω)).

The θ(ε$_{\Omega+1}$^2) level separator

{a, b [1 [1 [1~3] 2 ~3] 2] 2} = {a ‹0 [1 [1~3] 2 ~3] 2› b}
= {a ‹S_1› b}.

Since p_1 = 1, c_1 = 2, $\#_1$ = #* = ‘’ (blank),
$[A_{1,1}]$ = [1 [1~3] 2 ~3]   (1-hyperseparator),
p_2 = 1, c_2 = 2, $\#_2$ = ‘~3’,
$[A_{2,1}]$ = [1~3]   (1-hyperseparator),
p_3 = 1, c_3 = 3, $\#_3$ = ‘’ (blank),
$[A_{3,1}]$ = ~ (= $/_2$)   (2-hyperseparator),

by Rule A5b (m = 1),
S_1 = ‘b ‹S_2› b’,
t_1 = 1, t_2 = 0;

by Rule A5b (m = 1),
S_2 = ‘b ‹S_3› b ~3’,
t_1 = 2, t_2 = 0;

and by Rule A5a (m = 2, s = 3),
S_3 = ‘$R_{b,3}$’,
$R_{n,3}$ = ‘b ‹$R_{n-1,3}$› b ~2’,
$R_{1,3}$ = ‘0’.

It follows that,

{a, b [1 [1 [1~3] 2 ~3] 2] 2} = {a ‹b ‹b ‹b ‹ ... ‹b ‹b~2› b~2› ... › b~2› b ~3› b› b}
(with b+1 pairs of angle brackets and b ~’s).

The ordinal levels of some higher separators are as follows:

[1 [1 [1~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}$^2),
[1 [1 [1~3] 3 ~3] 2] has level θ(ε$_{\Omega+1}$^3),
[1 [1 [1~3] 1 [1 [1~3] 2] 2 ~3] 2] has level θ(ε$_{\Omega+1}$^θ(ε$_{\Omega+1}$)),
[1 [1 [1~3] 1 [1 [1 [1~3] 1 [1 [1~3] 2] 2 ~3] 2] 2 ~3] 2] has level θ(ε$_{\Omega+1}$^θ(ε$_{\Omega+1}$^θ(ε$_{\Omega+1}$))),
[1 [1 [1~3] 1 / 2 ~3] 2] has level θ(ε$_{\Omega+1}$^Ω),
[1 [1 [1~3] 1 / 3 ~3] 2] has level θ(ε$_{\Omega+1}$^(Ω2)),
[1 [1 [1~3] 1 / 1 / 2 ~3] 2] has level θ(ε$_{\Omega+1}$^Ω^2),
[1 [1 [1~3] 1 [2~2] 2 ~3] 2] has level θ(ε$_{\Omega+1}$^Ω^ω),
[1 [1 [1~3] 1 [1 / 2 ~2] 2 ~3] 2] has level θ(ε$_{\Omega+1}$^Ω^Ω),
[1 [1 [1~3] 1 [1 / 1 / 2 ~2] 2 ~3] 2] has level θ(ε$_{\Omega+1}$^Ω^Ω^Ω),
[1 [1 [1~3] 1 [1 [1 / 2 ~2] 2 ~2] 2 ~3] 2] has level θ(ε$_{\Omega+1}$^Ω^Ω^Ω^Ω),
[1 [1 [1~3] 1 [1~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$),
[1 [1 [1~3] 1 [1~3] 3 ~3] 2] has level θ(ε$_{\Omega+1}$^(ε$_{\Omega+1}$2)),
[1 [1 [1~3] 1 [1~3] 1 [1~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$^2),
[1 [1 [1~3] 1 [1~3] 1 [1~3] 1 [1~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$^3),
[1 [1 [2~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$^ω),
[1 [1 [1 [1 [1 [1~3] 1 [1~3] 2 ~3] 2] 2 ~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$^θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$)),
[1 [1 [1 / 2 ~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$^Ω),
[1 [1 [1 / 1 / 2 ~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$^Ω^Ω),

[1 [1 [1 [1 / 2 ~2] 2 ~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$^Ω^Ω^Ω),
[1 [1 [1 [1 / 1 / 2 ~2] 2 ~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$^Ω^Ω^Ω^Ω),
[1 [1 [1 [1 [1 / 2 ~2] 2 ~2] 2 ~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}$^Ω^Ω^Ω^Ω^Ω),
[1 [1 [1 [1~3] 2 ~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}^{ε}_{\Omega+1}$),
[1 [1 [1 [1~3] 1 [1~3] 2 ~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}^{ε}_{\Omega+1}^{ε}_{\Omega+1}$),
[1 [1 [1 [1 [1~3] 2 ~3] 2 ~3] 2 ~3] 2] has level θ(ε$_{\Omega+1}^{ε}_{\Omega+1}^{ε}_{\Omega+1}^{ε}_{\Omega+1}^{ε}_{\Omega+1}$).

[1 [1~4] 2] has level θ(ε$_{\Omega+2}$),
[1 [1~5] 2] has level θ(ε$_{\Omega+3}$)   (by letting '~3' and ε$_{\Omega+1}$ above be '~4' and ε$_{\Omega+2}$ respectively),
[1 [1 ~ 1, 2] 2] has level θ(ε$_{\Omega+\omega}$),
[1 [1 ~ 1 [1 [1~3] 2] 2] 2] has level θ(ε(Ω+θ(ε$_{\Omega+1}$))),
[1 [1 ~ 1 [1 [1 ~ 1 [1 [1~3] 2] 2] 2] 2] 2] has level θ(ε(Ω+θ(ε(Ω+θ(ε$_{\Omega+1}$))))),
[1 [1 ~ 1 / 2] 2] has level θ(ε$_{\Omega2}$),
[1 [1 ~ 2 / 2] 2] has level θ(ε$_{\Omega2+1}$)
(by letting '~3' and ε$_{\Omega+1}$ above be '~ 1 / 2' and ε$_{\Omega2}$ respectively),
[1 [1 ~ 1 [1 [1 ~ 1 / 2] 2] 2 / 2] 2] has level θ(ε(Ω2+θ(ε$_{\Omega2}$))),
[1 [1 ~ 1 [1 [1 ~ 1 [1 [1 ~ 1 / 2] 2] 2 / 2] 2] 2 / 2] 2] has level θ(ε(Ω2+θ(ε(Ω2+θ(ε$_{\Omega2}$))))),
[1 [1 ~ 1 / 3] 2] has level θ(ε$_{\Omega3}$),
[1 [1 ~ 1 / 4] 2] has level θ(ε$_{\Omega4}$),
[1 [1 ~ 1 / 1 [1 [1~3] 2] 2] 2] has level θ(ε(Ωθ(ε$_{\Omega+1}$))),
[1 [1 ~ 1 / 1 [1 [1 ~ 1 / 1 [1 [1~3] 2] 2] 2] 2] 2] has level θ(ε(Ωθ(ε(Ωθ(ε$_{\Omega+1}$))))),
[1 [1 ~ 1 / 1 / 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}2}$),
[1 [1 ~ 1 / 1 / 1 / 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}3}$),
[1 [1 ~ 1 [2~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\omega}$),
[1 [1 ~ 1 [1 [1 [1~3] 2] 2 ~2] 2] 2] has level θ(ε(Ω^θ(ε$_{\Omega+1}$))),
[1 [1 ~ 1 [1 [1 [1 ~ 1 [1 [1 [1~3] 2] 2 ~2] 2] 2] 2 ~2] 2] 2] has level θ(ε(Ω^θ(ε(Ω^θ(ε$_{\Omega+1}$))))).

[1 [1 ~ 1 [1 / 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega}$),
[1 [1 ~ 2 [1 / 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega+1}$),
[1 [1 ~ 1 / 2 [1 / 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega+\Omega}$),
[1 [1 ~ 1 [1 / 2 ~2] 3] 2] has level θ(ε$_{(\Omega\text{\textasciicircum}\Omega)2}$),
[1 [1 ~ 1 [1 / 2 ~2] 1 / 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}(\Omega+1)}$),
[1 [1 ~ 1 [1 / 2 ~2] 1 [1 / 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}(\Omega2)}$),
[1 [1 ~ 1 [2 / 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}(\Omega\omega)}$),
[1 [1 ~ 1 [1 [1 [1 ~ 1 [1 / 2 ~2] 2] 2] 2 / 2 ~2] 2] 2] has level θ(ε(Ω^(Ωθ(ε$_{\Omega\text{\textasciicircum}\Omega}$)))),
[1 [1 ~ 1 [1 / 3 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}2}$),
[1 [1 ~ 1 [1 / 1 [1 [1 ~ 1 [1 / 2 ~2] 2] 2] 2 ~2] 2] 2] has level θ(ε(Ω^Ω^θ(ε$_{\Omega\text{\textasciicircum}\Omega}$))),
[1 [1 ~ 1 [1 / 1 / 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega}$),
[1 [1 ~ 1 [1 / 1 / 1 / 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}2}$),
[1 [1 ~ 1 [1 [2~2] 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\omega}$),
[1 [1 ~ 1 [1 [1 [1 [1 ~ 1 [1 / 1 / 2 ~2] 2] 2] 2 ~2] 2 ~2] 2] 2] has level θ(ε(Ω^Ω^Ω^θ(ε$_{\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega}$))),
[1 [1 ~ 1 [1 [1 / 2 ~2] 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega}$),
[1 [1 ~ 1 [1 [1 / 1 / 2 ~2] 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega}$),
[1 [1 ~ 1 [1 [1 [1 / 2 ~2] 2 ~2] 2 ~2] 2] 2] has level θ(ε$_{\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega\text{\textasciicircum}\Omega}$).

[1 [1 ~ 1 [1~3] 2] 2] has level θ(ε(ε$_{\Omega+1}$)),
[1 [1 ~ 2 [1~3] 2] 2] has level θ(ε(ε$_{\Omega+1}$+1)),
[1 [1 ~ 1 / 2 [1~3] 2] 2] has level θ(ε(ε$_{\Omega+1}$+Ω)),
[1 [1 ~ 1 [1~3] 3] 2] has level θ(ε(ε$_{\Omega+1}$2)),

[1 [1 ~ 1 [1~3] 1 / 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}\Omega))$,
[1 [1 ~ 1 [1~3] 1 [1~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{2}))$,
[1 [1 ~ 1 [2~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{\omega}))$,
[1 [1 ~ 1 [1 / 2 ~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{\Omega}))$,
[1 [1 ~ 1 [1 / 1 / 2 ~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{\Omega^{\Omega}}))$,
[1 [1 ~ 1 [1 [1 / 2 ~2] 2 ~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{\Omega^{\Omega^{\Omega}}}))$,
[1 [1 ~ 1 [1 [1 / 1 / 2 ~2] 2 ~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{\Omega^{\Omega^{\Omega^{\Omega}}}}))$,
[1 [1 ~ 1 [1 [1 [1 / 2 ~2] 2 ~2] 2 ~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{\Omega^{\Omega^{\Omega^{\Omega^{\Omega}}}}}))$,
[1 [1 ~ 1 [1 [1 / 2 ~3] 2 ~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{\varepsilon_{\Omega+1}}))$,
[1 [1 ~ 1 [1 [1 / 1 / 2 ~3] 2 ~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{\varepsilon_{\Omega+1}^{\varepsilon_{\Omega+1}}}))$,
[1 [1 ~ 1 [1 [1 [1 / 2 ~3] 2 ~3] 2 ~3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+1}^{\varepsilon_{\Omega+1}^{\varepsilon_{\Omega+1}^{\varepsilon_{\Omega+1}}}}))$,
[1 [1 ~ 1 [1~4] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega+2}))$,
[1 [1 ~ 1 [1 ~ 1 / 2] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega 2}))$,
[1 [1 ~ 1 [1 ~ 1 / 3] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega 3}))$,
[1 [1 ~ 1 [1 ~ 1 / 1 / 2] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega^{2}}))$,
[1 [1 ~ 1 [1 ~ 1 / 1 / 1 / 2] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega^{3}}))$,
[1 [1 ~ 1 [1 ~ 1 [2~2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega^{\omega}}))$,
[1 [1 ~ 1 [1 ~ 1 [1 / 2 ~2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega^{\Omega}}))$,
[1 [1 ~ 1 [1 ~ 1 [1 / 1 / 2 ~2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega^{\Omega^{\Omega}}}))$,
[1 [1 ~ 1 [1 ~ 1 [1 [1 / 2 ~2] 2 ~2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega^{\Omega^{\Omega^{\Omega}}}}))$,
[1 [1 ~ 1 [1 ~ 1 [1 [1 / 1 / 2 ~2] 2 ~2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega^{\Omega^{\Omega^{\Omega^{\Omega}}}}}))$,
[1 [1 ~ 1 [1 ~ 1 [1 [1 [1 / 2 ~2] 2 ~2] 2 ~2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon_{\Omega^{\Omega^{\Omega^{\Omega^{\Omega^{\Omega}}}}}}))$.

[1 [1 ~ 1 [1 ~ 1 [1~3] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1})))$,
[1 [1 ~ 1 [1 ~ 1 [1~4] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+2})))$,
[1 [1 ~ 1 [1 ~ 1 [1 ~ 1 / 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega 2})))$,
[1 [1 ~ 1 [1 ~ 1 [1 ~ 1 / 1 / 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega^{2}})))$,
[1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1 / 2 ~2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega^{\Omega}})))$,
[1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1 / 1 / 2 ~2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega^{\Omega^{\Omega}}})))$,
[1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1 [1 / 2 ~2] 2 ~2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega^{\Omega^{\Omega^{\Omega}}}})))$,
[1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1~3] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}))))$,
[1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1~3] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+1})))))$,
[1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1~3] 2] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}))))))$.

[1 [1 ~ 1 ~ 2] 2] has level $\theta(\zeta_{\Omega+1}) = \theta(\varphi(2, \Omega+1))$,
[1 [1 ~ 1 ~ 2] 1 [1 ~ 1 ~ 2] 2] has level $\theta(\zeta_{\Omega+1}2)$,
[1 [2 ~ 1 ~ 2] 2] has level $\theta(\zeta_{\Omega+1}\omega)$,
[1 [1 / 2 ~ 1 ~ 2] 2] has level $\theta(\zeta_{\Omega+1}\Omega)$,
[1 [1 [1~3] 2 ~ 1 ~ 2] 2] has level $\theta(\zeta_{\Omega+1}\varepsilon_{\Omega+1})$,
[1 [1 [1~1~2] 2 ~ 1 ~ 2] 2] has level $\theta(\zeta_{\Omega+1}^{2})$,
[1 [1 [1~1~2] 1 / 2 ~ 1 ~ 2] 2] has level $\theta(\zeta_{\Omega+1}^{\Omega})$,
[1 [1 [1~1~2] 1 [1~3] 2 ~ 1 ~ 2] 2] has level $\theta(\zeta_{\Omega+1}^{\varepsilon_{\Omega+1}})$,
[1 [1 [1~1~2] 1 [1~1~2] 2 ~ 1 ~ 2] 2] has level $\theta(\zeta_{\Omega+1}^{\zeta_{\Omega+1}})$,
[1 [1 [1 [1~1~2] 2 ~1~2] 2 ~1~2] 2] has level $\theta(\zeta_{\Omega+1}^{\zeta_{\Omega+1}^{\zeta_{\Omega+1}}})$,
[1 [1 [1 [1~1~2] 1 [1~1~2] 2 ~1~2] 2 ~1~2] 2] has level $\theta(\zeta_{\Omega+1}^{\zeta_{\Omega+1}^{\zeta_{\Omega+1}^{\zeta_{\Omega+1}}}})$,
[1 [1 [1 [1 [1~1~2] 2 ~1~2] 2 ~1~2] 2 ~1~2] 2] has level $\theta(\zeta_{\Omega+1}^{\zeta_{\Omega+1}^{\zeta_{\Omega+1}^{\zeta_{\Omega+1}^{\zeta_{\Omega+1}}}}})$.

[1 [1 ~ 2 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}+1))$,
[1 [1 ~ 3 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}+2))$,

[1 [1 ~ 1 / 2 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}+\Omega))$,
[1 [1 ~ 1 [1~1~2] 2 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}2))$,
[1 [1 ~ 1 [1~1~2] 1 / 2 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}\Omega))$,
[1 [1 ~ 1 [1~1~2] 1 [1~1~2] 2 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}{}^\wedge 2))$,
[1 [1 ~ 1 [1 / 2 ~1~2] 2 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}{}^\wedge\Omega))$,
[1 [1 ~ 1 [1 [1~1~2] 2 ~1~2] 2 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}{}^\wedge\zeta_{\Omega+1}))$,
[1 [1 ~ 1 [1 [1~1~2] 1 [1~1~2] 2 ~1~2] 2 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}{}^\wedge\zeta_{\Omega+1}{}^\wedge\zeta_{\Omega+1}))$,
[1 [1 ~ 1 [1 [1 [1~1~2] 2 ~1~2] 2 ~1~2] 2 ~ 2] 2] has level $\theta(\varepsilon(\zeta_{\Omega+1}{}^\wedge\zeta_{\Omega+1}{}^\wedge\zeta_{\Omega+1}{}^\wedge\zeta_{\Omega+1}))$,
[1 [1 ~ 1 [1~2~2] 2 ~ 2] 2] has level $\theta(\varepsilon(\varepsilon(\zeta_{\Omega+1}+1)))$,
[1 [1 ~ 1 [1 ~ 1 [1~2~2] 2 ~ 2] 2 ~ 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\zeta_{\Omega+1}+1))))$,
[1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1~2~2] 2 ~ 2] 2 ~ 2] 2 ~ 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\varepsilon(\zeta_{\Omega+1}+1)))))$.

[1 [1 ~ 1 ~ 3] 2] has level $\theta(\zeta_{\Omega+2})$,
[1 [1 ~ 1 ~ 4] 2] has level $\theta(\zeta_{\Omega+3})$,
[1 [1 ~ 1 ~ 1 / 2] 2] has level $\theta(\zeta_{\Omega 2})$,
[1 [1 ~ 1 ~ 1 / 1 / 2] 2] has level $\theta(\zeta_{\Omega^\wedge 2})$,
[1 [1 ~ 1 ~ 1 [2~2] 2] 2] has level $\theta(\zeta_{\Omega^\wedge\omega})$,
[1 [1 ~ 1 ~ 1 [1 / 2 ~2] 2] 2] has level $\theta(\zeta_{\Omega^\wedge\Omega})$,
[1 [1 ~ 1 ~ 1 [1 / 1 / 2 ~2] 2] 2] has level $\theta(\zeta_{\Omega^\wedge\Omega^\wedge\Omega})$,
[1 [1 ~ 1 ~ 1 [1 [1 / 2 ~2] 2 ~2] 2] 2] has level $\theta(\zeta_{\Omega^\wedge\Omega^\wedge\Omega^\wedge\Omega})$,
[1 [1 ~ 1 ~ 1 [1~3] 2] 2] has level $\theta(\zeta(\varepsilon_{\Omega+1}))$,
[1 [1 ~ 1 ~ 1 [1 ~ 1 / 2] 2] 2] has level $\theta(\zeta(\varepsilon_{\Omega 2}))$,
[1 [1 ~ 1 ~ 1 [1 ~ 1 / 1 / 2] 2] 2] has level $\theta(\zeta(\varepsilon_{\Omega^\wedge 2}))$,
[1 [1 ~ 1 ~ 1 [1 ~ 1 [1 / 2 ~2] 2] 2] 2] has level $\theta(\zeta(\varepsilon_{\Omega^\wedge\Omega}))$,
[1 [1 ~ 1 ~ 1 [1 ~ 1 [1 / 1 / 2 ~2] 2] 2] 2] has level $\theta(\zeta(\varepsilon_{\Omega^\wedge\Omega^\wedge\Omega}))$,
[1 [1 ~ 1 ~ 1 [1 ~ 1 [1 [1 / 2 ~2] 2 ~2] 2] 2] 2] has level $\theta(\zeta(\varepsilon_{\Omega^\wedge\Omega^\wedge\Omega^\wedge\Omega}))$,
[1 [1 ~ 1 ~ 1 [1 ~ 1 [1~3] 2] 2] 2] has level $\theta(\zeta(\varepsilon(\varepsilon_{\Omega+1})))$,
[1 [1 ~ 1 ~ 1 [1 ~ 1 [1 ~ 1 [1~3] 2] 2] 2] 2] has level $\theta(\zeta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}))))$,
[1 [1 ~ 1 ~ 1 [1 ~ 1 [1 ~ 1 [1 ~ 1 [1~3] 2] 2] 2] 2] 2] has level $\theta(\zeta(\varepsilon(\varepsilon(\varepsilon(\varepsilon_{\Omega+1})))))$,
[1 [1 ~ 1 ~ 1 [1~1~2] 2] 2] has level $\theta(\zeta(\zeta_{\Omega+1}))$,
[1 [1 ~ 1 ~ 1 [1~1~ 1 [1~1~2] 2] 2] 2] has level $\theta(\zeta(\zeta(\zeta_{\Omega+1})))$,
[1 [1 ~ 1 ~ 1 [1~1~ 1 [1~1~ 1 [1~1~2] 2] 2] 2] 2] has level $\theta(\zeta(\zeta(\zeta(\zeta_{\Omega+1}))))$.

[1 [1 ~ 1 ~ 1 ~ 2] 2] has level $\theta(\varphi(3, \Omega+1))$,
[1 [2 ~ 1 ~ 1 ~ 2] 2] has level $\theta(\varphi(3, \Omega+1)\omega)$,
[1 [1 / 2 ~ 1 ~ 1 ~ 2] 2] has level $\theta(\varphi(3, \Omega+1)\Omega)$,
[1 [1 [1~1~1~2] 2 ~ 1 ~ 1 ~ 2] 2] has level $\theta(\varphi(3, \Omega+1)^\wedge 2)$,
[1 [1 [1~1~1~2] 1 [1~1~1~2] 2 ~1~1~2] 2] has level $\theta(\varphi(3, \Omega+1)^\wedge\varphi(3, \Omega+1))$,
[1 [1 [1 [1~1~1~2] 2 ~1~1~2] 2 ~1~1~2] 2] has level $\theta(\varphi(3, \Omega+1)^\wedge\varphi(3, \Omega+1)^\wedge\varphi(3, \Omega+1))$,
[1 [1 ~ 2 ~ 1 ~ 2] 2] has level $\theta(\varepsilon_{\varphi(3, \Omega+1)+1})$,
[1 [1 ~ 1 [1~2~1~2] 2 ~ 1 ~ 2] 2] has level $\theta(\varepsilon(\varepsilon_{\varphi(3, \Omega+1)+1}))$,
[1 [1 ~ 1 ~ 2 ~ 2] 2] has level $\theta(\zeta_{\varphi(3, \Omega+1)+1})$,
[1 [1 ~ 1 ~ 1 [1~1~2~2] 2 ~ 2] 2] has level $\theta(\zeta(\zeta_{\varphi(3, \Omega+1)+1}))$,
[1 [1 ~ 1 ~ 1 ~ 3] 2] has level $\theta(\varphi(3, \Omega+2))$,
[1 [1 ~ 1 ~ 1 ~ 1 / 2] 2] has level $\theta(\varphi(3, \Omega 2))$,
[1 [1 ~ 1 ~ 1 ~ 1 / 1 / 2] 2] has level $\theta(\varphi(3, \Omega^\wedge 2))$,
[1 [1 ~ 1 ~ 1 ~ 1 [1 / 2 ~2] 2] 2] has level $\theta(\varphi(3, \Omega^\wedge\Omega))$,
[1 [1 ~ 1 ~ 1 ~ 1 [1~3] 2] 2] has level $\theta(\varphi(3, \varepsilon_{\Omega+1}))$,
[1 [1 ~ 1 ~ 1 ~ 1 [1~1~2] 2] 2] has level $\theta(\varphi(3, \zeta_{\Omega+1}))$,

[1 [1 ~ 1 ~ 1 ~ 1 [1~1~1~2] 2] 2] has level θ(φ(3, φ(3, Ω+1))),
[1 [1 ~ 1 ~ 1 ~ 1 [1~1~1~ 1 [1~1~1~2] 2] 2] 2] 2] has level θ(φ(3, φ(3, φ(3, Ω+1)))),
[1 [1 ~ 1 ~ 1 ~ 1 ~ 2] 2] has level θ(φ(4, Ω+1)),
[1 [1 ~ 1 ~ 1 ~ 1 ~ 1 ~ 2] 2] has level θ(φ(5, Ω+1)),
[1 [1 ~ 1 ~ 1 ~ ... ~ 1 ~ 2] 2] (with n ~ symbols, n ≥ 2) has level θ(φ(n, Ω+1)).

Separators containing the $/_3$ symbol ([1 $/_3$ 2] 'drops down' to $/_2$ or ~) begin as follows:
[1 [1 [2 $/_3$ 2] 2] 2] has level θ(φ(ω, Ω+1)),
[1 [2 [2 $/_3$ 2] 2] 2] has level θ(φ(ω, Ω+1)ω),
[1 [1 / 2 [2 $/_3$ 2] 2] 2] has level θ(φ(ω, Ω+1)Ω),
[1 [1 [1 [2 $/_3$ 2] 2] 2 [2 $/_3$ 2] 2] 2] has level θ(φ(ω, Ω+1)^2),
[1 [1 [1 [2 $/_3$ 2] 2] 1 [1 [2 $/_3$ 2] 2] 2 [2 $/_3$ 2] 2] 2] has level θ(φ(ω, Ω+1)^φ(ω, Ω+1)),
[1 [1 [1 [1 [2 $/_3$ 2] 2] 2 [2 $/_3$ 2] 2] 2 [2 $/_3$ 2] 2] 2] has level θ(φ(ω, Ω+1)^^3),
[1 [1 ~ 2 [2 $/_3$ 2] 2] 2] has level θ($\varepsilon_{\varphi(\omega, \Omega+1)+1}$),
[1 [1 ~ 1 [1 ~ 2 [2 $/_3$ 2] 2] 2 [2 $/_3$ 2] 2] 2] has level θ(ε($\varepsilon_{\varphi(\omega, \Omega+1)+1}$)),
[1 [1 ~ 1 ~ 2 [2 $/_3$ 2] 2] 2] has level θ($\zeta_{\varphi(\omega, \Omega+1)+1}$),
[1 [1 ~ 1 ~ 1 ~ 2 [2 $/_3$ 2] 2] 2] has level θ(φ(3, φ(ω, Ω+1)+1)),
[1 [1 ~ 1 ~ ... ~ 1 ~ 2 [2 $/_3$ 2] 2] 2] (with n ~ symbols) has level θ(φ(n, φ(ω, Ω+1)+1)),
[1 [1 [2 $/_3$ 2] 3] 2] has level θ(φ(ω, Ω+2))   (limit ordinal of φ(n, φ(ω, Ω+1)+1) as n → ω),
[1 [1 [2 $/_3$ 2] 4] 2] has level θ(φ(ω, Ω+3)),
[1 [1 [2 $/_3$ 2] 1 / 2] 2] has level θ(φ(ω, Ω2)),
[1 [1 [2 $/_3$ 2] 1 / 1 / 2] 2] has level θ(φ(ω, Ω^2)),
[1 [1 [2 $/_3$ 2] 1 [1 / 2 ~2] 2] 2] has level θ(φ(ω, Ω^Ω)),
[1 [1 [2 $/_3$ 2] 1 [1~3] 2] 2] has level θ(φ(ω, $\varepsilon_{\Omega+1}$)),
[1 [1 [2 $/_3$ 2] 1 [1~1~2] 2] 2] has level θ(φ(ω, $\zeta_{\Omega+1}$)),
[1 [1 [2 $/_3$ 2] 1 [1~1~...~1~2] 2] 2] (with n ~ symbols, n ≥ 2) has level θ(φ(ω, φ(n, Ω+1))),
[1 [1 [2 $/_3$ 2] 1 [1 [2 $/_3$ 2] 2] 2] 2] has level θ(φ(ω, φ(ω, Ω+1))),
[1 [1 [2 $/_3$ 2] 1 [1 [2 $/_3$ 2] 1 [1 [2 $/_3$ 2] 2] 2] 2] 2] has level θ(φ(ω, φ(ω, φ(ω, Ω+1)))).

[1 [1 [2 $/_3$ 2] 1 ~ 2] 2] has level θ(φ(ω+1, Ω+1)),
[1 [1 [2 $/_3$ 2] 1 ~ 1 ~ 2] 2] has level θ(φ(ω+2, Ω+1)),
[1 [1 [2 $/_3$ 2] 1 [2 $/_3$ 2] 2] 2] has level θ(φ(ω2, Ω+1)),
[1 [1 [2 $/_3$ 2] 1 [2 $/_3$ 2] 1 [2 $/_3$ 2] 2] 2] has level θ(φ(ω3, Ω+1)),
[1 [1 [3 $/_3$ 2] 2] 2] has level θ(φ(ω^2, Ω+1)),
[1 [1 [1, 2 $/_3$ 2] 2] 2] has level θ(φ(ω^ω, Ω+1)),
[1 [1 [1 [1 / 2] 2 $/_3$ 2] 2] 2] has level θ(φ(ε_0, Ω+1)),
[1 [1 [1 [1 [1 / 2 ~2] 2] 2 $/_3$ 2] 2] 2] has level θ(φ(Γ_0, Ω+1)),
[1 [1 [1 [1 [1~3] 2] 2 $/_3$ 2] 2] 2] has level θ(φ(θ($\varepsilon_{\Omega+1}$), Ω+1)),
[1 [1 [1 [1 [1 [1 [1 [1~3] 2] 2 $/_3$ 2] 2] 2] 2 $/_3$ 2] 2] 2] has level θ(φ(θ(φ(θ($\varepsilon_{\Omega+1}$), Ω+1)), Ω+1)),
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1 [1~3] 2] 2 $/_3$ 2] 2] 2] 2 $/_3$ 2] 2] 2] 2 $/_3$ 2] 2] 2] has level
θ(φ(θ(φ(θ(φ(θ($\varepsilon_{\Omega+1}$), Ω+1)), Ω+1)), Ω+1)).

The sequence of separators starting with the last three has limit ordinal θ(φ(Ω, 1)). This is due to the fact that φ(Ω, 1) is the limit of φ(α, Ω+1) = φ(α, φ(Ω, 0)+1) as α → Ω.

A second (more powerful) ordinal collapsing function (θ_1) can be used within the θ function in order to represent the ordinals above the Bachmann-Howard ordinal. It works as follows:
$\theta_1(0, \alpha) = \Omega$^α,
$\theta_1(1, \alpha) = \varepsilon_{\Omega+1+\alpha}$,
$\theta_1(2, \alpha) = \zeta_{\Omega+1+\alpha}$,

$\theta_1(\alpha) = \theta_1(\alpha, 0)$,

$\theta_1(\alpha, \beta) = \varphi(\alpha, \Omega+1+\beta)$ $\quad (1 \le \alpha < \Omega)$

$\quad = \varphi(\alpha, 1+\beta)$ $\quad (\alpha = \Omega)$

$\quad = \varphi(\alpha, \beta)$ $\quad (\Omega < \alpha < \Omega_2)$,

$\theta_1(\alpha+1) = \theta_1(\alpha, \theta_1(\alpha, \theta_1(\alpha, \ldots \theta_1(\alpha)\ldots)))$ $\quad$ (with ω θ_1's),

$\theta_1(\Omega) = \varphi(\Omega, 1) = \theta_1(\theta(\theta_1(\theta(\theta_1(\ldots\theta(\theta_1(0))\ldots)))))$ $\quad$ (with ω θ_1's),

$\theta_1(\Omega_2) = \Gamma_{\Omega+1} = \theta_1(\theta_1(\theta_1(\ldots\theta_1(0)\ldots)))$ $\quad$ (with ω θ_1's),

$\theta(\theta_1(\alpha, \beta)) = \theta(\alpha, \beta)$ $\quad (\alpha \ge \Omega_2)$,

where Ω_2 denotes the second uncountable ordinal. ($\Omega = \Omega_1$ is the first uncountable ordinal.) We can define higher collapsing functions (θ_n) and create higher uncountable ordinals (Ω_{n+1}) by analogy with θ_1 and Ω_2 above, for example,

$\theta_n(0, \alpha) = \Omega_n\text{^}\alpha$,

$\theta_n(\alpha) = \theta_n(\alpha, 0)$,

$\theta_n(\alpha, \beta) = \varphi(\alpha, \Omega_n+1+\beta)$ $\quad (1 \le \alpha < \Omega_n)$

$\quad = \varphi(\alpha, 1+\beta)$ $\quad (\alpha = \Omega_n)$

$\quad = \varphi(\alpha, \beta)$ $\quad (\Omega_n < \alpha < \Omega_{n+1})$,

$\theta_n(\alpha+1) = \theta_n(\alpha, \theta_n(\alpha, \theta_n(\alpha, \ldots \theta_1(\alpha)\ldots)))$ $\quad$ (with ω θ_n's),

$\theta_n(\Omega) = \theta_n(\theta(\theta_n(\theta(\theta_n(\ldots\theta(\theta_n(0))\ldots)))))$ $\quad$ (with ω θ_n's),

$\theta_n(\Omega_{k+1}) = \theta_n(\theta_k(\theta_n(\theta_k(\theta_n(\ldots\theta_k(\theta_n(0))\ldots)))))$ $\quad$ (with ω θ_n's, $k \le n$),

$\theta_n(\Omega_{n+1}) = \theta_n(\theta_n(\theta_n(\ldots\theta_n(0)\ldots)))$ $\quad$ (with ω θ_n's),

$\theta(\theta_n(\alpha, \beta)) = \theta(\alpha, \beta)$ $\quad (\alpha \ge \Omega_{n+1})$,

$\theta_k(\theta_n(\alpha, \beta)) = \theta_k(\alpha, \beta)$ $\quad (\alpha \ge \Omega_{n+1}, k < n)$.

We can extend this to θ_α functions and Ω_α uncountable ordinals for transfinite α. $\theta(\Omega_\omega)$ is a special ordinal since it is the proof theoretic ordinal of the subsystem Π^1_1-CA_0 of second-order arithmetic. There are whole new universes of ordinals completely beyond the first fixed point of $\alpha = \Omega_\alpha$ within θ.

An alternative (single-argument) ordinal collapsing function (ψ) works as follows:

$\psi(\alpha) = \varepsilon_\alpha$ $\quad (\alpha < \Omega)$,

$\psi(\alpha+1) = \psi(\alpha)\text{^^}\omega$ $\quad$ (power tower of $\psi(\alpha)$'s of height ω),

$\psi((\Omega\text{^}\alpha)\beta) = \varphi(1+\alpha, \beta-1)$ $\quad (1 \le \alpha < \Omega, 1 \le \beta < \omega)$

$\quad = \varphi(1+\alpha, \beta)$ $\quad (1 \le \alpha < \Omega, \omega \le \beta < \Omega)$

$\quad = \theta(1+\alpha, \beta-1)$ $\quad (\alpha \ge 1, 1 \le \beta < \omega)$

$\quad = \theta(1+\alpha, \beta)$ $\quad (\alpha \ge 1, \omega \le \beta < \Omega)$.

Like the θ_n functions (within θ), we can define ψ_n functions (within ψ, for $n \ge 1$) in order to proceed beyond the Bachmann-Howard ordinal. It works as follows:

$\psi_n(\alpha) = \varepsilon(\Omega_n+1+\alpha)$ $\quad (\alpha < \Omega_{n+1})$,

$\psi_n(\alpha+1) = \psi_n(\alpha)\text{^^}\omega$ $\quad$ (power tower of $\psi_n(\alpha)$'s of height ω),

$\psi_n((\Omega_{n+1}\text{^}\alpha)\beta) = \varphi(1+\alpha, \Omega_n+\beta)$ $\quad (1 \le \alpha < \Omega_n, 1 \le \beta < \Omega_{n+1})$

$\quad = \varphi(\alpha, \beta-1)$ $\quad (\Omega_n < \alpha < \Omega_{n+1}, 1 \le \beta < \omega)$

$\quad = \varphi(\alpha, \beta)$ $\quad (\Omega_n < \alpha < \Omega_{n+1}, \omega \le \beta < \Omega_{n+1}$ or $\alpha = \Omega_n, 1 \le \beta < \Omega_{n+1})$

$\quad = \theta_n(1+\alpha, \beta-1)$ $\quad (\alpha \ge 1, 1 \le \beta < \omega)$

$\quad = \theta_n(1+\alpha, \beta)$ $\quad (\alpha \ge 1, \omega \le \beta < \Omega_{n+1})$,

$\psi(\psi_n(\alpha)) = \psi(\alpha)$ $\quad (\alpha \ge \Omega_{n+1})$,

$\psi_k(\psi_n(\alpha)) = \psi_k(\alpha)$ $\quad (\alpha \ge \Omega_{n+1}, k < n)$.

This can be extended further by defining higher ψ_α functions for transfinite α. I prefer to use the θ system of collapsing functions since these begin with the finite numbers (rather than ε_0) and it is more closely related to the Veblen function. Also, a power tower of Ω's within the θ function contains one fewer Ω than the equivalent power tower of Ω's within the ψ function.

The ordinal $\theta(\varphi(\Omega, 1)) = \theta(\theta_1(\Omega)) = \psi_1(\Omega_2\hat{}\Omega)$, using the θ_1 and ψ_1 functions as defined above.

Using the θ_1 function, the most important separators at or beyond the level of the Bachmann-Howard ordinal are as follows:

[1 [1~3] 2] has level $\theta(\theta_1(1))$,
[1 [1~3] 1 / 2] has level $\theta(\theta_1(1)+1)$,
[1 [2~3] 2] has level $\theta(\theta_1(1)\omega)$,
[1 [1 / 2 ~3] 2] has level $\theta(\theta_1(1)\Omega)$,
[1 [1 [1~3] 2 ~3] 2] has level $\theta(\theta_1(1)\hat{}2)$,
[1 [1 [1~3] 1 / 2 ~3] 2] has level $\theta(\theta_1(1)\hat{}\Omega)$,
[1 [1 [1~3] 1 [1~3] 2 ~3] 2] has level $\theta(\theta_1(1)\hat{}\theta_1(1))$,
[1 [1 [1 [1~3] 2 ~3] 2 ~3] 2] has level $\theta(\theta_1(1)\hat{}\theta_1(1)\hat{}\theta_1(1))$,
[1 [1~4] 2] has level $\theta(\theta_1(1, 1))$,
[1 [1 ~ 1 / 2] 2] has level $\theta(\theta_1(1, \Omega))$,
[1 [1 ~ 1 / 1 / 2] 2] has level $\theta(\theta_1(1, \Omega\hat{}2))$,
[1 [1 ~ 1 [1 / 2 ~2] 2] 2] has level $\theta(\theta_1(1, \Omega\hat{}\Omega))$,
[1 [1 ~ 1 [1~3] 2] 2] has level $\theta(\theta_1(1, \theta_1(1)))$,
[1 [1 ~ 1 [1 ~ 1 [1~3] 2] 2] 2] has level $\theta(\theta_1(1, \theta_1(1, \theta_1(1))))$,
[1 [1 ~ 1 ~ 2] 2] has level $\theta(\theta_1(2))$,
[1 [1 ~ 2 ~ 2] 2] has level $\theta(\theta_1(1, \theta_1(2)+1))$,
[1 [1 ~ 1 ~ 3] 2] has level $\theta(\theta_1(2, 1))$,
[1 [1 ~ 1 ~ 1 / 2] 2] has level $\theta(\theta_1(2, \Omega))$,
[1 [1 ~ 1 ~ 1 / 1 / 2] 2] has level $\theta(\theta_1(2, \Omega\hat{}2))$,
[1 [1 ~ 1 ~ 1 [1 / 2 ~2] 2] 2] has level $\theta(\theta_1(2, \Omega\hat{}\Omega))$,
[1 [1 ~ 1 ~ 1 [1~3] 2] 2] has level $\theta(\theta_1(2, \theta_1(1)))$,
[1 [1 ~ 1 ~ 1 [1~1~2] 2] 2] has level $\theta(\theta_1(2, \theta_1(2)))$,
[1 [1 ~ 1 ~ 1 ~ 2] 2] has level $\theta(\theta_1(3))$,
[1 [1 [2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\omega))$,
[1 [1 [1 [1 / 2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\varepsilon_0))$,
[1 [1 [1 [1 [1~3] 2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta(\theta_1(1))))$,
[1 [1 [1 [1 [1 [1 [1 [1~3] 2] 2 $/_3$ 2] 2] 2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta(\theta_1(\theta(\theta_1(1))))))$.

Higher separators are obtained as follows:

[1 [1 [1 / 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega)) = \theta(\varphi(\Omega, 1))$,
[1 [1 [1 / 2 $/_3$ 2] 3] 2] has level $\theta(\theta_1(\Omega, 1)) = \theta(\varphi(\Omega, 2))$,
[1 [1 [1 / 2 $/_3$ 2] 1 [1 [1 / 2 $/_3$ 2] 2] 2] 2] has level $\theta(\theta_1(\Omega, \theta_1(\Omega)))$,
[1 [1 [1 / 2 $/_3$ 2] 1 ~ 2] 2] has level $\theta(\theta_1(\Omega+1))$,
[1 [1 [1 / 2 $/_3$ 2] 1 [1 / 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega2))$,
[1 [1 [2 / 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\omega))$,
[1 [1 [1 [1 [1 [1 / 2 $/_3$ 2] 2] 2] 2 / 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\theta(\theta_1(\Omega))))$,
[1 [1 [1 / 3 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\hat{}2))$,
[1 [1 [1 / 1 / 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\hat{}\Omega))$,
[1 [1 [1 / 1 / 1 / 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\hat{}\Omega\hat{}2))$,
[1 [1 [1 [2~2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\hat{}\Omega\hat{}\omega))$,
[1 [1 [1 [1 [1 [1 [1 / 1 / 2 $/_3$ 2] 2] 2] 2 ~2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\hat{}\Omega\hat{}\theta(\theta_1(\Omega\hat{}\Omega))))$,
[1 [1 [1 [1 / 2 ~2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\hat{}\Omega\hat{}\Omega))$,
[1 [1 [1 [1 / 1 / 2 ~2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\hat{}\Omega\hat{}\Omega\hat{}\Omega))$,
[1 [1 [1 [1 [1 / 2 ~2] 2 ~2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega\hat{}\Omega\hat{}\Omega\hat{}\Omega\hat{}\Omega))$,
[1 [1 [1 [1~3] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta_1(1))) = \theta(\theta_1(\varepsilon_{\Omega+1}))$,

[1 [1 [1 [1~4] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta_1(1, 1))) = \theta(\theta_1(\varepsilon_{\Omega+2}))$,
[1 [1 [1 [1~1~2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta_1(2))) = \theta(\theta_1(\zeta_{\Omega+1}))$,
[1 [1 [1 [1 [2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta_1(\omega)))$,
[1 [1 [1 [1 [1 [1 / 2] 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta_1(\varepsilon_0)))$,
[1 [1 [1 [1 [1 [1 [1~3] 2] 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta_1(\theta(\theta_1(1))))) = \theta(\theta_1(\theta_1(\theta(\varepsilon_{\Omega+1}))))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1~3] 2] 2 $/_3$ 2] 2] 2] 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2] has level
$\theta(\theta_1(\theta_1(\theta(\theta_1(\theta(\theta_1(1)))))))$,
[1 [1 [1 [1 [1 / 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta_1(\Omega)))$,
[1 [1 [1 [1 [1 [1~3] 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta_1(\theta_1(1))))$,
[1 [1 [1 [1 [1 [1 [1 [1~3] 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\theta_1(\theta_1(\theta_1(1)))))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1~3] 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2 $/_3$ 2] 2] 2] has level
$\theta(\theta_1(\theta_1(\theta_1(\theta_1(\theta_1(1))))))$.

The sequence of separators starting with the last three has limit ordinal $\theta(\Omega_2) = \theta(\theta_1(\Omega_2)) = \theta(\Gamma_{\Omega+1})$, where Ω_2 denotes the second uncountable ordinal.

The $\theta(\theta_1(\varepsilon_{\Omega+1}))$ level separator
{a, b [1 [1 [1 [1~3] 2 $/_3$ 2] 2] 2] 2} = {a ‹0 [1 [1 [1~3] 2 $/_3$ 2] 2] 2› b}
= {a ‹S_1› b}.

Since $p_1 = 1$, $c_1 = 2$, $\#_1 = \#^* =$ ‘’ (blank),
[$A_{1,1}$] = [1 [1 [1~3] 2 $/_3$ 2] 2] (1-hyperseparator),
$p_2 = 1$, $c_2 = 2$, $\#_2 =$ ‘’ (blank),
[$A_{2,1}$] = [1 [1~3] 2 $/_3$ 2] (2-hyperseparator),
$p_3 = 1$, $c_3 = 2$, $\#_3 =$ ‘$/_3$ 2’,
[$A_{3,1}$] = [1~3] (1-hyperseparator),
$p_4 = 1$, $c_4 = 3$, $\#_4 =$ ‘’ (blank),
[$A_{4,1}$] = ~ (= $/_2$) (2-hyperseparator),

by Rule A5b (m = 1),
S_1 = ‘b ‹S_2› b’,
$t_1 = 1$, $t_2 = 0$, $t_3 = 0$;

by Rule A5b (m = 2),
S_2 = ‘b ‹S_3› b’,
$t_1 = 2$, $t_2 = 1$, $t_3 = 0$;

by Rule A5b (m = 1),
S_3 = ‘b ‹S_4› b $/_3$ 2’,
$t_1 = 3$, $t_2 = 0$, $t_3 = 0$;

and by Rule A5a (m = 2, s = 4),
S_4 = ‘$R_{b,4}$’,
$R_{n,4}$ = ‘b ‹$R_{n-1,4}$› b ~2’,
$R_{1,4}$ = ‘0’.

It follows that,
{a, b [1 [1 [1 [1~3] 2 $/_3$ 2] 2] 2] 2}
= {a ‹b ‹b ‹b ‹b ‹b ‹ ... ‹b ‹b~2› b~2› ... › b~2› b~2› b $/_3$ 2› b› b› b}
(with b+2 pairs of angle brackets and b-1 ~’s).

The $\theta(\Omega_2)$ level separator
{a, b [1 [1 [1 ~ 2 $/_3$ 2] 2] 2] 2} = {a ‹0 [1 [1 ~ 2 $/_3$ 2] 2] 2› b}
= {a ‹S_1› b}.

Since $p_1 = 1$, $c_1 = 2$, $\#_1 = \#^* =$ ‘’ (blank),
[$A_{1,1}$] = [1 [1 ~ 2 $/_3$ 2] 2] (1-hyperseparator),

$p_2 = 1$, $c_2 = 2$, $\#_2$ = ‘’ (blank),
$[A_{2,1}]$ = [1 ~ 2 $/_3$ 2] (2-hyperseparator),
$p_3 = 1$, $c_3 = 2$, $\#_3$ = ‘$/_3$ 2’,
$[A_{3,1}]$ = ~ (= $/_2$) (2-hyperseparator),

by Rule A5b (m = 1),
S_1 = ‘b ‹S_2› b’,
$t_1 = 1$, $t_2 = 0$, $t_3 = 0$;

by Rule A5b (m = 2),
S_2 = ‘b ‹S_3› b’,
$t_1 = 2$, $t_2 = 1$, $t_3 = 0$;

and by Rule A5a (m = 2, s = 2),
S_3 = ‘$R_{b,3}$’,
$R_{n,3}$ = ‘b ‹$R_{n-1,2}$› b $/_3$ 2’,
$R_{n,2}$ = ‘b ‹$R_{n,3}$› b’ (so, $S_2 = R_{b,2}$)
= ‘b ‹b ‹$R_{n-1,2}$› b $/_3$ 2› b’,
$R_{1,2}$ = ‘0’.

It follows that,
{a, b [1 [1 [1 ~ 2 $/_3$ 2] 2] 2] 2}
= {a ‹b ‹b ‹b ‹b ‹b ‹ ... ‹b ‹b ‹b ‹b $/_3$ 2› b› b $/_3$ 2› b› ... › b $/_3$ 2› b› b $/_3$ 2› b› b› b}
(with 2b-1 pairs of angle brackets and b-1 $/_3$’s).

$\theta(\theta_1(\alpha, \beta)) = \theta(\alpha, \beta)$ for all $\alpha \ge \Omega_2$, $\beta < \Omega_2$ – the θ function has been redefined to absorb the θ_1 function for values of $\alpha \ge \Omega_2$. With this redefinition, the second argument of θ now has a limit value of Ω_2 instead of Ω, for values of $\alpha \ge \Omega_2$. Continuing with the separators,

[1 [1 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2)$,
[1 [1 [1 ~ 2 $/_3$ 2] 2] 3] has level $\theta(\theta_1(\Omega_2), 1)$,
[1 [1 [1 ~ 2 $/_3$ 2] 2] 1 [1 [1 [1 ~ 2 $/_3$ 2] 2] 2] 2] has level $\theta(\theta_1(\Omega_2), \theta(\Omega_2))$,
[1 [1 [1 ~ 2 $/_3$ 2] 2] 1 / 2] has level $\theta(\theta_1(\Omega_2)+1)$,
[1 [1 [1 ~ 2 $/_3$ 2] 2] 1 [1~3] 2] has level $\theta(\theta_1(\Omega_2)+\varepsilon_{\Omega+1}) = \theta(\theta_1(\Omega_2)+\theta_1(1))$,
[1 [1 [1 ~ 2 $/_3$ 2] 2] 1 [1 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega_2)2)$,
[1 [2 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega_2)\omega)$,
[1 [1 / 2 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega_2)\Omega)$,
[1 [1 [1~3] 2 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega_2)\varepsilon_{\Omega+1}) = \theta(\theta_1(\Omega_2)\theta_1(1))$,
[1 [1 [1 [1 ~ 2 $/_3$ 2] 2] 2 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega_2)\text{^}2)$,
[1 [1 ~ 2 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\varepsilon(\theta_1(\Omega_2)+1))$,
[1 [1 [2 $/_3$ 2] 2 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\omega, \theta_1(\Omega_2)+1))$,
[1 [1 [1 / 2 $/_3$ 2] 2 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\theta_1(\Omega, \theta_1(\Omega_2)+1))$,
[1 [1 [1 ~ 2 $/_3$ 2] 3] 2] has level $\theta(\Omega_2, 1) = \theta(\theta_1(\Omega_2, 1))$
($\theta_1(\Omega_2, 1)$ is the limit of $\theta_1(\alpha, \theta_1(\Omega_2)+1)$ as $\alpha \to \Omega_2$),
[1 [1 [1 ~ 2 $/_3$ 2] 1 [1 [1 [1 ~ 2 $/_3$ 2] 2] 2] 2] 2] has level $\theta(\Omega_2, \theta(\Omega_2))$,
[1 [1 [1 ~ 2 $/_3$ 2] 1 / 2] 2] has level $\theta(\Omega_2, \Omega)$,
[1 [1 [1 ~ 2 $/_3$ 2] 1 [1 [1 ~ 2 $/_3$ 2] 2] 2] 2] has level $\theta(\Omega_2, \theta_1(\Omega_2))$,
[1 [1 [1 ~ 2 $/_3$ 2] 1 ~ 2] 2] has level $\theta(\Omega_2+1)$,
[1 [1 [1 ~ 2 $/_3$ 2] 1 [1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2 2)$,
[1 [1 [2 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2\omega)$,
[1 [1 [1 [1 [1 [1 ~ 2 $/_3$ 2] 2] 2] 2 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2\theta(\Omega_2))$,
[1 [1 [1 / 2 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2\Omega)$,
[1 [1 [1 / 1 / 2 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2(\Omega\text{^}\Omega))$,
[1 [1 [1 [1 / 2 ~2] 2 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2(\Omega\text{^}\Omega\text{^}\Omega))$,

[1 [1 [1 [1~3] 2 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2\varepsilon_{\Omega+1}) = \theta(\Omega_2\theta_1(1))$,
[1 [1 [1 [1 [1 / 2 $/_3$ 2] 2] 2 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2\theta_1(\Omega))$,
[1 [1 [1 [1 [1 ~ 2 $/_3$ 2] 2] 2 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2\theta_1(\Omega_2))$,
[1 [1 [1 ~ 3 $/_3$ 2] 2] 2] has level $\theta(\Omega_2$^2),
[1 [1 [1 ~ 1 / 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2$^Ω),
[1 [1 [1 ~ 1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2$^$\Omega_2)$,
[1 [1 [1 ~ 1 ~ 1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2$^Ω_2^2),
[1 [1 [1 [2 $/_3$ 2] 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2$^Ω_2^ω),
[1 [1 [1 [1 / 2 $/_3$ 2] 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2$^Ω_2^Ω),
[1 [1 [1 [1 ~ 2 $/_3$ 2] 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2$^Ω_2^$\Omega_2)$,
[1 [1 [1 [1 ~ 1 ~ 2 $/_3$ 2] 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2$^Ω_2^Ω_2^$\Omega_2)$,
[1 [1 [1 [1 [1 ~ 2 $/_3$ 2] 2 $/_3$ 2] 2 $/_3$ 2] 2] 2] has level $\theta(\Omega_2$^Ω_2^Ω_2^Ω_2^$\Omega_2)$.

[1 [1 [1 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)) = \theta(\theta_2(1))$,
[1 [1 [1 $/_3$ 3] 3] 2] has level $\theta(\varepsilon(\Omega_2+1), 1)$,
[1 [1 [1 $/_3$ 3] 1 / 2] 2] has level $\theta(\varepsilon(\Omega_2+1), \Omega)$,
[1 [1 [1 $/_3$ 3] 1 [1 [1 $/_3$ 3] 2] 2] 2] has level $\theta(\varepsilon(\Omega_2+1), \theta_1(\varepsilon(\Omega_2+1)))$,
[1 [1 [1 $/_3$ 3] 1 ~ 2] 2] has level $\theta(\varepsilon(\Omega_2+1)+1)$,
[1 [1 [1 $/_3$ 3] 1 [1 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)2)$,
[1 [1 [2 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)\omega)$,
[1 [1 [1 / 2 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)\Omega)$,
[1 [1 [1 [1 [1 $/_3$ 3] 2] 2 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)\theta_1(\varepsilon(\Omega_2+1)))$,
[1 [1 [1 ~ 2 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)\Omega_2)$,
[1 [1 [1 [1 $/_3$ 3] 2 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)$^2),
[1 [1 [1 [1 $/_3$ 3] 1 [1 $/_3$ 3] 2 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)$^$\varepsilon(\Omega_2+1))$,
[1 [1 [1 [1 [1 $/_3$ 3] 2 $/_3$ 3] 2 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)$^$\varepsilon(\Omega_2+1)$^$\varepsilon(\Omega_2+1))$,
[1 [1 [1 $/_3$ 4] 2] 2] has level $\theta(\varepsilon(\Omega_2+2)) = \theta(\theta_2(1, 1))$,
[1 [1 [1 $/_3$ 1 / 2] 2] 2] has level $\theta(\varepsilon(\Omega_2+\Omega)) = \theta(\theta_2(1, \Omega))$,
[1 [1 [1 $/_3$ 1 ~ 2] 2] 2] has level $\theta(\varepsilon(\Omega_2 2)) = \theta(\theta_2(1, \Omega_2))$,
[1 [1 [1 $/_3$ 1 ~ 1 ~ 2] 2] 2] has level $\theta(\varepsilon(\Omega_2$^2)),
[1 [1 [1 $/_3$ 1 [1 / 2 $/_3$ 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_2$^Ω)),
[1 [1 [1 $/_3$ 1 [1 ~ 2 $/_3$ 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_2$^$\Omega_2))$,
[1 [1 [1 $/_3$ 1 [1 ~ 1 ~ 2 $/_3$ 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_2$^$\Omega_2$^$\Omega_2))$,
[1 [1 [1 $/_3$ 1 [1 [1 ~ 2 $/_3$ 2] 2 $/_3$ 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_2$^$\Omega_2$^$\Omega_2$^$\Omega_2))$,
[1 [1 [1 $/_3$ 1 [1 $/_3$ 3] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\Omega_2+1)))$,
[1 [1 [1 $/_3$ 1 [1 $/_3$ 1 [1 $/_3$ 3] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\Omega_2+1))))$.

[1 [1 [1 $/_3$ 1 $/_3$ 2] 2] 2] has level $\theta(\zeta(\Omega_2+1)) = \theta(\theta_2(2))$,
[1 [1 [1 / 2 $/_3$ 1 $/_3$ 2] 2] 2] has level $\theta(\zeta(\Omega_2+1)\Omega)$,
[1 [1 [1 ~ 2 $/_3$ 1 $/_3$ 2] 2] 2] has level $\theta(\zeta(\Omega_2+1)\Omega_2)$,
[1 [1 [1 [1 $/_3$ 1 $/_3$ 2] 2 $/_3$ 1 $/_3$ 2] 2] 2] has level $\theta(\zeta(\Omega_2+1)$^2),
[1 [1 [1 $/_3$ 2 $/_3$ 2] 2] 2] has level $\theta(\varepsilon(\zeta(\Omega_2+1)+1))$,
[1 [1 [1 $/_3$ 1 / 2 $/_3$ 2] 2] 2] has level $\theta(\varepsilon(\zeta(\Omega_2+1)+\Omega))$,
[1 [1 [1 $/_3$ 1 ~ 2 $/_3$ 2] 2] 2] has level $\theta(\varepsilon(\zeta(\Omega_2+1)+\Omega_2))$,
[1 [1 [1 $/_3$ 1 [1 $/_3$ 1 $/_3$ 2] 2 $/_3$ 2] 2] 2] has level $\theta(\varepsilon(\zeta(\Omega_2+1)2))$,
[1 [1 [1 $/_3$ 1 [1 $/_3$ 2 $/_3$ 2] 2 $/_3$ 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\zeta(\Omega_2+1)+1)))$,
[1 [1 [1 $/_3$ 1 $/_3$ 3] 2] 2] has level $\theta(\zeta(\Omega_2+2)) = \theta(\theta_2(2, 1))$,
[1 [1 [1 $/_3$ 1 $/_3$ 1 / 2] 2] 2] has level $\theta(\zeta(\Omega_2+\Omega)) = \theta(\theta_2(2, \Omega))$,
[1 [1 [1 $/_3$ 1 $/_3$ 1 ~ 2] 2] 2] has level $\theta(\zeta(\Omega_2 2)) = \theta(\theta_2(2, \Omega_2))$,

[1 [1 [1 $/_3$ 1 $/_3$ 1 ~ 1 ~ 2] 2] 2] has level $\theta(\zeta(\Omega_2\hat{}2))$,
[1 [1 [1 $/_3$ 1 $/_3$ 1 [1 ~ 2 $/_3$ 2] 2] 2] 2] has level $\theta(\zeta(\Omega_2\hat{}\Omega_2))$,
[1 [1 [1 $/_3$ 1 $/_3$ 1 [1 $/_3$ 3] 2] 2] 2] has level $\theta(\zeta(\varepsilon(\Omega_2+1)))$,
[1 [1 [1 $/_3$ 1 $/_3$ 1 [1 $/_3$ 1 $/_3$ 2] 2] 2] 2] has level $\theta(\zeta(\zeta(\Omega_2+1)))$,
[1 [1 [1 $/_3$ 1 $/_3$ 1 $/_3$ 2] 2] 2] has level $\theta(\theta_2(3)) = \theta(\varphi(3, \Omega_2+1))$,
[1 [1 [1 $/_3$ 1 $/_3$ 1 $/_3$ 1 $/_3$ 2] 2] 2] has level $\theta(\theta_2(4))$.

Separators containing the $/_4$ symbol ([1 $/_4$ 2] 'drops down' to $/_3$) begin as follows:
[1 [1 [1 [2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\omega))$,
[1 [1 [1 [2 $/_4$ 2] 1 $/_3$ 2] 2] 2] has level $\theta(\theta_2(\omega+1))$,
[1 [1 [1 [2 $/_4$ 2] 1 [2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\omega 2))$,
[1 [1 [1 [3 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\omega\hat{}2))$,
[1 [1 [1 [1, 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\omega\hat{}\omega))$,
[1 [1 [1 [1 [1 / 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\varepsilon_0))$,
[1 [1 [1 [1 [1 [1 [1 $/_3$ 3] 2] 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta(\theta_2(1))))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1 [1 $/_3$ 3] 2] 2] 2 $/_4$ 2] 2] 2] 2] 2 $/_4$ 2] 2] 2] 2] has level
$\theta(\theta_2(\theta(\theta_2(\theta(\theta_2(1))))))$,
[1 [1 [1 [1 / 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\Omega))$,
[1 [1 [1 [1 [1~3] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_1(1)))$,
[1 [1 [1 [1 [1 [1 / 2 $/_3$ 2] 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_1(\Omega)))$,
[1 [1 [1 [1 [1 [1 ~ 2 $/_3$ 2] 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_1(\Omega_2)))$,
[1 [1 [1 [1 [1 [1 $/_3$ 3] 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_1(\theta_2(1))))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 $/_3$ 3] 2] 2 $/_4$ 2] 2] 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_1(\theta_2(\theta_1(\theta_2(1))))))$,
[1 [1 [1 [1 ~ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\Omega_2))$,
[1 [1 [1 [1 [1 $/_3$ 3] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_2(1)))$,
[1 [1 [1 [1 [1 [1 / 2 $/_4$ 2] 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_2(\Omega)))$,
[1 [1 [1 [1 [1 [1 ~ 2 $/_4$ 2] 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_2(\Omega_2)))$,
[1 [1 [1 [1 [1 [1 [1 $/_3$ 3] 2 $/_4$ 2] 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_2(\theta_2(1))))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 $/_3$ 3] 2 $/_4$ 2] 2] 2 $/_4$ 2] 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\theta_2(\theta_2(\theta_2(1)))))$.

The sequence of separators starting with the last two has limit ordinal $\theta(\Omega_3) = \theta(\theta_2(\Omega_3)) = \theta(\Gamma(\Omega_2+1))$, where Ω_3 denotes the third uncountable ordinal.

The $\theta(\theta_2(\Omega_k))$ level separator (k = 1, 2 or 3)
{a, b [1 [1 [1 [1 $/_k$ 2 $/_4$ 2] 2] 2] 2] 2} = {a ‹0 [1 [1 [1 $/_k$ 2 $/_4$ 2] 2] 2] 2› b}
= {a ‹S_1› b}.

Since $p_1 = 1$, $c_1 = 2$, $\#_1 = \#^* =$ '' (blank),
[$A_{1,1}$] = [1 [1 [1 $/_k$ 2 $/_4$ 2] 2] 2] (1-hyperseparator),
$p_2 = 1$, $c_2 = 2$, $\#_2 =$ '' (blank),
[$A_{2,1}$] = [1 [1 $/_k$ 2 $/_4$ 2] 2] (2-hyperseparator),
$p_3 = 1$, $c_3 = 2$, $\#_3 =$ '' (blank),
[$A_{3,1}$] = [1 $/_k$ 2 $/_4$ 2] (3-hyperseparator),
$p_4 = 1$, $c_4 = 2$, $\#_4 =$ '$/_4$ 2',
[$A_{4,1}$] = $/_k$ (k-hyperseparator),

by Rule A5b (m = 1),
S_1 = 'b ‹S_2› b',
$t_1 = 1$, $t_2 = 0$, $t_3 = 0$, $t_4 = 0$;

by Rule A5b (m = 2),
S_2 = 'b ‹S_3› b',
$t_1 = 2$, $t_2 = 1$, $t_3 = 0$, $t_4 = 0$;

by Rule A5b (m = 3),

S_3 = ‘b ‹S_4› b’,

$t_1 = 3$, $t_2 = 2$, $t_3 = 1$, $t_4 = 0$;

and by Rule A5a (m = k, s = 4-t_k = k; the k in the $R_{n,k}$ string is renamed j),

S_4 = ‘$R_{b,4}$’,

$R_{n,4}$ = ‘b ‹$R_{n-1,k}$› b $/_4$ 2’,

$R_{n,j}$ = ‘b ‹$R_{n,j+1}$› b’ (k ≤ j < 4; so, $S_k = R_{b,k}$),

$R_{1,k}$ = ‘0’.

It follows that,

{a, b [1 [1 [1 [1 $/_k$ 2 $/_4$ 2] 2] 2] 2] 2}

= {a ‹b ‹b ‹b ‹b ‹b ‹b ‹b ‹b ‹ ... ‹b ‹b ‹b ‹b ‹b ‹b $/_4$ 2› b› b› b› b $/_4$ 2› b› ... › b $/_4$ 2› b› b› b› b $/_4$ 2› b› b› b› b}
(with 4b-4 pairs of angle brackets and b-1 $/_4$’s, k = 1)

= {a ‹b ‹b ‹b ‹b ‹b ‹b ‹b ‹ ... ‹b ‹b ‹b ‹b ‹b $/_4$ 2› b› b› b $/_4$ 2› b› ... › b $/_4$ 2› b› b› b $/_4$ 2› b› b› b› b}
(with 3b-2 pairs of angle brackets and b-1 $/_4$’s, k = 2)

= {a ‹b ‹b ‹b ‹b ‹b ‹b ‹ ... ‹b ‹b ‹b ‹b $/_4$ 2› b› b $/_4$ 2› b› ... › b $/_4$ 2› b› b $/_4$ 2› b› b› b› b}
(with 2b pairs of angle brackets and b-1 $/_4$’s, k = 3).

$\theta(\theta_2(\alpha, \beta)) = \theta(\alpha, \beta)$ and $\theta_1(\theta_2(\alpha, \beta)) = \theta_1(\alpha, \beta)$ for all $\alpha \geq \Omega_3$, $\beta < \Omega_3$ – the θ and θ_1 functions have been redefined to absorb the θ_2 function for values of $\alpha \geq \Omega_3$. With this redefinition, the second arguments of θ and θ_1 now have a limit value of Ω_3 for values of $\alpha \geq \Omega_3$. Continuing with the separators,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3)$,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 3] has level $\theta(\theta_1(\Omega_3), 1)$ (as $\theta_1(\Omega_3) = \theta_1(\theta_2(\Omega_3))$),

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 1 / 2] has level $\theta(\theta_1(\Omega_3)+1)$,

[1 [2 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_1(\Omega_3)\omega)$,

[1 [1 / 2 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_1(\Omega_3)\Omega)$,

[1 [1 ~ 2 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\varepsilon(\theta_1(\Omega_3)+1))$,

[1 [1 [1 [1 / 2 $/_4$ 2] 2] 2 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_1(\Omega, \theta_1(\Omega_3)+1))$,

[1 [1 [1 [1 ~ 2 $/_4$ 2] 2] 2 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_1(\Omega_2, \theta_1(\Omega_3)+1))$,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 3] 2] has level $\theta(\theta_2(\Omega_3), 1) = \theta(\theta_1(\theta_2(\Omega_3), 1))$

($\theta_1(\theta_2(\Omega_3), 1)$ is the limit of $\theta_1(\alpha, \theta_1(\theta_2(\Omega_3))+1)$ as $\alpha \rightarrow \theta_2(\Omega_3)$),

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 1 / 2] 2] has level $\theta(\theta_2(\Omega_3), \Omega)$,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 1 ~ 2] 2] has level $\theta(\theta_2(\Omega_3)+1)$,

[1 [1 [2 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\Omega_3)\omega)$,

[1 [1 [1 / 2 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\Omega_3)\Omega)$,

[1 [1 [1 ~ 2 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\Omega_3)\Omega_2)$,

[1 [1 [1 $/_3$ 2 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\varepsilon(\theta_2(\Omega_3)+1))$,

[1 [1 [1 [1 / 2 $/_4$ 2] 2 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\Omega, \theta_2(\Omega_3)+1))$,

[1 [1 [1 [1 ~ 2 $/_4$ 2] 2 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_2(\Omega_2, \theta_2(\Omega_3)+1))$,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 3] 2] 2] has level $\theta(\Omega_3, 1) = \theta(\theta_2(\Omega_3, 1))$

($\theta_2(\Omega_3, 1)$ is the limit of $\theta_2(\alpha, \theta_2(\Omega_3)+1)$ as $\alpha \rightarrow \Omega_3$),

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 1 [1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] 2] 2] 2] has level $\theta(\Omega_3, \theta(\Omega_3))$,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 1 / 2] 2] 2] has level $\theta(\Omega_3, \Omega)$,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] 2] 2] has level $\theta(\Omega_3, \theta_1(\Omega_3))$,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 1 ~ 2] 2] 2] has level $\theta(\Omega_3, \Omega_2)$,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] 2] has level $\theta(\Omega_3, \theta_2(\Omega_3))$,

[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 1 $/_3$ 2] 2] 2] has level $\theta(\Omega_3+1)$,

[1 [1 [1 [2 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3\omega)$,

[1 [1 [1 [1 [1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] 2 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3\theta(\Omega_3))$,

[1 [1 [1 [1 / 2 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3\Omega)$,
[1 [1 [1 [1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3\theta_1(\Omega_3))$,
[1 [1 [1 [1 ~ 2 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3\Omega_2)$,
[1 [1 [1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3\theta_2(\Omega_3))$,
[1 [1 [1 [1 $/_3$ 3 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3{}^\wedge 2)$,
[1 [1 [1 [1 $/_3$ 1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3{}^\wedge\Omega_3)$,
[1 [1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3{}^\wedge\Omega_3{}^\wedge\Omega_3)$,
[1 [1 [1 [1 [1 $/_3$ 1 $/_3$ 2 $/_4$ 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3{}^\wedge\Omega_3{}^\wedge\Omega_3{}^\wedge\Omega_3)$,
[1 [1 [1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2 $/_4$ 2] 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Omega_3{}^\wedge\Omega_3{}^\wedge\Omega_3{}^\wedge\Omega_3{}^\wedge\Omega_3)$.

[1 [1 [1 [1 $/_4$ 3] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1)) = \theta(\theta_3(1))$,
[1 [1 [1 [1 $/_4$ 3] 3] 2] 2] has level $\theta(\varepsilon(\Omega_3+1), 1)$,
[1 [1 [1 [1 $/_4$ 3] 1 / 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1), \Omega)$,
[1 [1 [1 [1 $/_4$ 3] 1 ~ 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1), \Omega_2)$,
[1 [1 [1 [1 $/_4$ 3] 1 $/_3$ 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1)+1)$,
[1 [1 [1 [2 $/_4$ 3] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1)\omega)$,
[1 [1 [1 [1 / 2 $/_4$ 3] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1)\Omega)$,
[1 [1 [1 [1 ~ 2 $/_4$ 3] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1)\Omega_2)$,
[1 [1 [1 [1 $/_3$ 2 $/_4$ 3] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1)\Omega_3)$,
[1 [1 [1 [1 [1 $/_4$ 3] 2 $/_4$ 3] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1){}^\wedge 2)$,
[1 [1 [1 [1 $/_4$ 4] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+2)) = \theta(\theta_3(1, 1))$,
[1 [1 [1 [1 $/_4$ 1 / 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+\Omega)) = \theta(\theta_3(1, \Omega))$,
[1 [1 [1 [1 $/_4$ 1 ~ 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+\Omega_2)) = \theta(\theta_3(1, \Omega_2))$,
[1 [1 [1 [1 $/_4$ 1 $/_3$ 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3 2)) = \theta(\theta_3(1, \Omega_3))$,
[1 [1 [1 [1 $/_4$ 1 $/_3$ 1 $/_3$ 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3{}^\wedge 2))$,
[1 [1 [1 [1 $/_4$ 1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3{}^\wedge\Omega_3))$,
[1 [1 [1 [1 $/_4$ 1 [1 $/_4$ 3] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\Omega_3+1)))$,
[1 [1 [1 [1 $/_4$ 1 [1 $/_4$ 1 [1 $/_4$ 3] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\varepsilon(\varepsilon(\Omega_3+1))))$,
[1 [1 [1 [1 $/_4$ 1 $/_4$ 2] 2] 2] 2] has level $\theta(\zeta(\Omega_3+1)) = \theta(\theta_3(2))$,
[1 [1 [1 [1 $/_4$ 1 $/_4$ 3] 2] 2] 2] has level $\theta(\zeta(\Omega_3+2)) = \theta(\theta_3(2, 1))$,
[1 [1 [1 [1 $/_4$ 1 $/_4$ 1 $/_3$ 2] 2] 2] 2] has level $\theta(\zeta(\Omega_3 2)) = \theta(\theta_3(2, \Omega_3))$,
[1 [1 [1 [1 $/_4$ 1 $/_4$ 1 [1 $/_4$ 1 $/_4$ 2] 2] 2] 2] 2] has level $\theta(\zeta(\zeta(\Omega_3+1)))$,
[1 [1 [1 [1 $/_4$ 1 $/_4$ 1 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_3(3)) = \theta(\varphi(3, \Omega_3+1))$,
[1 [1 [1 [1 $/_4$ 1 $/_4$ 1 $/_4$ 1 $/_4$ 2] 2] 2] 2] has level $\theta(\theta_3(4))$.

Separators containing the $/_5$ symbol ([1 $/_5$ 2] ‘drops down’ to $/_4$) begin as follows:
[1 [1 [1 [1 [2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\omega))$,
[1 [1 [1 [1 [1 [1 / 2] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\varepsilon_0))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 $/_4$ 3] 2] 2] 2] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\theta(\theta_3(1))))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1 [1 [1 [1 [1 $/_4$ 3] 2] 2] 2] 2 $/_5$ 2] 2] 2] 2] 2] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\theta(\theta_3(\theta(\theta_3(1))))))$,
[1 [1 [1 [1 [1 / 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\Omega))$,
[1 [1 [1 [1 [1 [1 [1 [1 $/_4$ 3] 2] 2] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\theta_1(\theta_3(1))))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1 [1 [1 $/_4$ 3] 2] 2] 2 $/_5$ 2] 2] 2] 2] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\theta_1(\theta_3(\theta_1(\theta_3(1))))))$,
[1 [1 [1 [1 [1 ~ 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\Omega_2))$,
[1 [1 [1 [1 [1 [1 [1 $/_4$ 3] 2] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\theta_2(\theta_3(1))))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1 $/_4$ 3] 2] 2 $/_5$ 2] 2] 2] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\theta_2(\theta_3(\theta_2(\theta_3(1))))))$,
[1 [1 [1 [1 [1 $/_3$ 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\Omega_3))$,

[1 [1 [1 [1 [1 [1 $/_4$ 3] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\theta_3(1)))$,
[1 [1 [1 [1 [1 [1 [1 [1 $/_4$ 3] 2 $/_5$ 2] 2] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\theta_3(\theta_3(1))))$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 [1 $/_4$ 3] 2 $/_5$ 2] 2] 2 $/_5$ 2] 2] 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\theta_3(\theta_3(\theta_3(\theta_3(1)))))$.

The separator sequence starting with the last three has limit ordinal $\theta(\Omega_4) = \theta(\theta_3(\Omega_4)) = \theta(\Gamma(\Omega_3+1))$, where Ω_4 denotes the fourth uncountable ordinal.

I have spotted a pattern in the development of the separators and their ordinal levels:

[1 / 2] has level $\varepsilon_0 = \theta(1)$,
[1 / 3] has level $\varepsilon_1 = \theta(1, 1)$,
[1 / 1 / 2] has level $\zeta_0 = \theta(2)$,
[1 [2 $/_2$ 2] 2] has level $\varphi(\omega, 0) = \theta(\omega)$,
[1 [1 / 2 $/_2$ 2] 2] has level $\Gamma_0 = \theta(\Omega)$,
[1 [1 $/_2$ 3] 2] has level $\theta(\varepsilon_{\Omega+1}) = \theta(\theta_1(1))$,
[1 [1 $/_2$ 1 $/_2$ 2] 2] has level $\theta(\zeta_{\Omega+1}) = \theta(\theta_1(2))$,
[1 [1 [2 $/_3$ 2] 2] 2] has level $\theta(\varphi(\omega, \Omega+1)) = \theta(\theta_1(\omega))$,
[1 [1 [1 $/_2$ 2 $/_3$ 2] 2] 2] has level $\theta(\Gamma_{\Omega+1}) = \theta(\Omega_2)$,
[1 [1 [1 $/_3$ 3] 2] 2] has level $\theta(\varepsilon(\Omega_2+1)) = \theta(\theta_2(1))$,
[1 [1 [1 $/_3$ 1 $/_3$ 2] 2] 2] has level $\theta(\zeta(\Omega_2+1)) = \theta(\theta_2(2))$,
[1 [1 [1 [2 $/_4$ 2] 2] 2] 2] has level $\theta(\varphi(\omega, \Omega_2+1)) = \theta(\theta_2(\omega))$,
[1 [1 [1 [1 $/_3$ 2 $/_4$ 2] 2] 2] 2] has level $\theta(\Gamma(\Omega_2+1)) = \theta(\Omega_3)$,
[1 [1 [1 [1 $/_4$ 3] 2] 2] 2] has level $\theta(\varepsilon(\Omega_3+1)) = \theta(\theta_3(1))$,
[1 [1 [1 [1 $/_4$ 1 $/_4$ 2] 2] 2] 2] has level $\theta(\zeta(\Omega_3+1)) = \theta(\theta_3(2))$,
[1 [1 [1 [1 [2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\varphi(\omega, \Omega_3+1)) = \theta(\theta_3(\omega))$.

Continuing this pattern, I would find that:

[1 [1 [1 [1 [1 $/_4$ 2 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\Gamma(\Omega_3+1)) = \theta(\Omega_4)$,
[1 [1 [1 [1 [1 $/_5$ 3] 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_4+1)) = \theta(\theta_4(1))$,
[1 [1 [1 [1 [1 $/_5$ 1 $/_5$ 2] 2] 2] 2] 2] has level $\theta(\zeta(\Omega_4+1)) = \theta(\theta_4(2))$,
[1 [1 [1 [1 [1 [2 $/_6$ 2] 2] 2] 2] 2] 2] has level $\theta(\varphi(\omega, \Omega_4+1)) = \theta(\theta_4(\omega))$,
[1 [1 [1 [1 [1 [1 $/_5$ 2 $/_6$ 2] 2] 2] 2] 2] 2] has level $\theta(\Gamma(\Omega_4+1)) = \theta(\Omega_5)$,
[1 [1 [1 [1 [1 [1 $/_6$ 3] 2] 2] 2] 2] 2] has level $\theta(\varepsilon(\Omega_5+1)) = \theta(\theta_5(1))$,
[1 [1 [1 [1 [1 [1 $/_6$ 1 $/_6$ 2] 2] 2] 2] 2] 2] has level $\theta(\zeta(\Omega_5+1)) = \theta(\theta_5(2))$,
[1 [1 [1 [1 [1 [1 [2 $/_7$ 2] 2] 2] 2] 2] 2] 2] has level $\theta(\varphi(\omega, \Omega_5+1)) = \theta(\theta_5(\omega))$,
[1 [1 [1 [1 [1 [1 [1 $/_6$ 2 $/_7$ 2] 2] 2] 2] 2] 2] 2] has level $\theta(\Gamma(\Omega_5+1)) = \theta(\Omega_6)$,
[1 [1 [1 [1 [1 [1 [1 [1 $/_7$ 2 $/_8$ 2] 2] 2] 2] 2] 2] 2] 2] has level $\theta(\Gamma(\Omega_6+1)) = \theta(\Omega_7)$,
[1 [1 [1 [1 [1 [1 [1 [1 [1 $/_8$ 2 $/_9$ 2] 2] 2] 2] 2] 2] 2] 2] 2] has level $\theta(\Gamma(\Omega_7+1)) = \theta(\Omega_8)$.

The sequence of separators starting with the last three has limit ordinal $\theta(\Omega_\omega)$, where Ω_ω denotes the limit of the nth uncountable ordinal (Ω_n) for finite n.

It can be seen that each additional nested layer of the separator (and increment of the forward slash subscript) requires a new uncountable ordinal within the θ function. Adding a generalised n-hyperseparator slash subscript symbol ($/_n$) somewhere in the separator array (or incrementing the number to the right of $/_n$ by 1) often generates an nth uncountable ordinal (Ω_n) somewhere in the associated ordinal expression, and these all combine together to produce Ω_{n+1} ordinals when we move up to $/_{n+1}$ symbols.

As the ordinals get larger, we redefine the $\theta, \theta_1, \theta_2, \ldots, \theta_{n-1}$ functions to absorb the θ_n function for values of $\alpha \geq \Omega_{n+1}$ by setting $\theta(\theta_n(\alpha, \beta)) = \theta(\alpha, \beta)$ and $\theta_k(\theta_n(\alpha, \beta)) = \theta_k(\alpha, \beta)$ for all $\alpha \geq \Omega_{n+1}$, $\beta < \Omega_{n+1}$ and $k < n$. This is to avoid writing ordinal expressions such as $\theta(\theta_1(\theta_2(\theta_3(\theta_4(\theta_5(\Omega_6))))))$. The second arguments of each of θ, θ_1, θ_2, ... , θ_{n-1} have a limit value of Ω_{n+1} for values of $\alpha \geq \Omega_{n+1}$. The θ function may be regarded as a θ_0 function in this respect.

In general, with n pairs of square brackets (n ≥ 3),

[1 [1 [... [1 [2 $/_n$ 2] 2] ...] 2] 2] has level $\theta(\varphi(\omega, \Omega_{n-2}+1)) = \theta(\theta_{n-2}(\omega))$,
[1 [1 [... [1 [1 $/_k$ 2 $/_n$ 2] 2] ...] 2] 2] has level $\theta(\theta_{n-2}(\Omega_k))$ (1 ≤ k < n),
[1 [1 [... [1 [1 $/_{n-1}$ 2 $/_n$ 2] 2] ...] 2] 2] has level $\theta(\Gamma(\Omega_{n-2}+1)) = \theta(\Omega_{n-1})$,
[1 [1 [... [1 [1 $/_n$ 3] 2] ...] 2] 2] has level $\theta(\varepsilon(\Omega_{n-1}+1)) = \theta(\theta_{n-1}(1))$,
[1 [1 [... [1 [1 $/_n$ 1 $/_n$ 2] 2] ...] 2] 2] has level $\theta(\zeta(\Omega_{n-1}+1)) = \theta(\theta_{n-1}(2))$.

With k+1 pairs of square brackets, the $\theta(\Omega_k)$ level separator

{a, b [1 [1 [... [1 [1 $/_k$ 2 $/_{k+1}$ 2] 2] ...] 2] 2] 2} = {a ‹0 [1 [... [1 [1 $/_k$ 2 $/_{k+1}$ 2] 2] ...] 2] 2› b}
= {a ‹S_1› b}.

Since $p_j = 1$, $c_j = 2$, $\#_j = \#^* =$ ‘’ (blank),
[$A_{j,1}$] = [1 [1 [... [1 [1 $/_k$ 2 $/_{k+1}$ 2] 2] ...] 2] 2]
(j-hyperseparator, with k-j+1 pairs of square brackets),

for 1 ≤ j ≤ k, and

$p_{k+1} = 1$, $c_{k+1} = 2$, $\#_{k+1}$ = ‘$/_{k+1}$ 2’,
[$A_{k+1,1}$] = $/_k$ (k-hyperseparator),

by the jth of k applications of Rule A5b (m = j, 1 ≤ j ≤ k),

S_j = ‘b ‹S_{j+1}› b’,
$t_1 = j$, $t_2 = j-1$, $t_3 = j-2$, ... , $t_j = 1$, $t_{j+1} = t_{j+2} = ... = t_{k+1} = 0$
(t-counters finish on $t_1 = k$, $t_2 = k-1$, $t_3 = k-2$, ... , $t_k = 1$, $t_{k+1} = 0$),

and by Rule A5a (m = k, s = k; the k in the $R_{n,k}$ string is also k),

S_{k+1} = ‘$R_{b,k+1}$’,
$R_{n,k+1}$ = ‘b ‹$R_{n-1,k}$› b $/_{k+1}$ 2’,
$R_{n,k}$ = ‘b ‹$R_{n,k+1}$› b’ (so, $S_k = R_{b,k}$)
= ‘b ‹b ‹$R_{n-1,k}$› b $/_{k+1}$ 2› b’,
$R_{1,k}$ = ‘0’.

It follows that,

{a, b [1 [1 [... [1 [1 $/_k$ 2 $/_{k+1}$ 2] 2] ...] 2] 2] 2} = {a ‹b ‹b ‹ ... ‹b ‹S› b› ... › b› b› b}
(with k pairs of angle brackets),

where S = ‘b ‹b ‹b ‹b ‹ ... ‹b ‹b ‹b ‹b $/_{k+1}$ 2› b› b $/_{k+1}$ 2› b› ... › b $/_{k+1}$ 2› b› b $/_{k+1}$ 2› b’
(with 2b-3 pairs of angle brackets and b-1 $/_{k+1}$’s).

The revised array notation, with modifications made to Angle Bracket Rule A5, backslashes turned into forward slashes and mixing of higher order hyperseparators of various levels on the same square bracket layer permitted (but otherwise similar to the Nested Hyper-Nested Array Notation in Beyond Bird’s Nested Arrays III), is called the Hierarchical Hyper-Nested Array Notation. The various levels of the hyperseparators are arranged in a strict hierarchy – not only does an (n+1)-hyperseparator outrank any n-hyperseparator but (n+1)-hyperseparators are only used when n-hyperseparators have been completely exhausted. While the hyperseparators in the previous paper were really all of the same level, with the backslash subscript $\backslash_n$ symbol simply representing the nth nested layer in the hyper-nested notation (in order to make it easier for readers to follow), the revised forward slash subscript $/_n$ symbol in this document represents any nested layer in the n-hyper-nested notation (or hyper-nested hyper-nested ... hyper-nested (n times) notation), so rather than christen it the n-Hyper-Nested Array Notation or ω-Hyper-Nested Array Notation, the name Hierarchical Hyper-Nested Array Notation is better as there is a hierarchy of sub-ε_0 level nested arrays, nested hyper-nested arrays, nested 2-hyper-nested arrays, nested 3-hyper-nested arrays, and so on.

I need not reproduce the Main Rules (M1-M7) here (when no angle brackets ‹ › appear in the main array), as these have remained completely unchanged from the Nested Array Notation (the Main Rules are on the first two pages of that paper). This is because the M rules only deal with the lowest layer of an entire curly bracket array, containing only entries separated by normal separators. Apart from Rule A5 and the direction of the slash symbols, none of the other Angle Bracket Rules have changed either, however, it is worth reproducing the revised complete Angle Bracket Rules, which are shown below.

Bird's Hierarchical Hyper-Nested Array Notation – Angle Bracket Rules

Rule A1 (only 1 entry of either 0 or 1):

‘a ‹0› b’ = ‘a’,

‘a ‹1› b’ = ‘a, a, ... , a’ (with b a's).

Rule A2 (only 1 entry of either 0 or 1 prior to 2-hyperseparator or higher order hyperseparator):

‘a ‹0 #› b’ = ‘a’,

‘a ‹1 #› b’ = ‘a [1 #] a [1 #] ... [1 #] a’ (with b a's),

where # begins with a 2- or higher order hyperseparator.

When n ≥ 2,

‘a ‹1 $/_n$ 2› b’ = ‘a $/_{n-1}$ a $/_{n-1}$ ... $/_{n-1}$ a’ (with b a's).

Rule A3 (last entry in any 1-space or higher dimensional space of array is 1):

‘a ‹# [A] 1› b’ = ‘a ‹#› b’.

When [A] is an m-hyperseparator, [B] is an n-hyperseparator and m < n, or m = n and level of [A] is less than level of [B],

‘a ‹# [A] 1 [B] #*› b’ = ‘a ‹# [B] #*› b’.

Remove trailing 1's.

Rule A4 (number to right of angle brackets is 1):

‘a ‹A› 1’ = ‘a’.

Rule A5 (Rules A1-4 do not apply, first entry is 0, separator immediately prior to next non-1 entry (c_1) is [A_{1,p_1}]):

‘a ‹ 0 [$A_{1,1}$] 1 [$A_{1,2}$] ... 1 [A_{1,p_1}] c_1 $\#_1$ #* › b’ = ‘a ‹S_1 #*› b’,

where $p_1 \geq 1$, each of [$A_{1,j}$] is either a normal separator or 1-hyperseparator, $\#_1$ contains no 2- or higher order hyperseparators in its base layer and #* is either an empty string or begins with a 2- or higher order hyperseparator.

Set i to 1 and t_1, t_2, ... , t_x to 0 (where x is the highest subscript to a forward slash within [A_{1,p_1}]), and follow Rules A5a-c (a and c are terminal, b is not). (Note that i = t_1+1 throughout.)

Rule A5a (separator [A_{i,p_i}] = [1 $/_{m+1}$ 2] = $/_m$, where m ≥ 1):

s = i-t_m,

S_i = ‘$R_{b,i}$’.

For n > 1 and s ≤ k < i,

$R_{n,i}$ = ‘b ‹$A_{i,1}$'› b [$A_{i,1}$] b ‹$A_{i,2}$'› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}'› b [A_{i,p_i-1}] b ‹$R_{n-1,s}$› b $/_m$ c_i-1 $\#_i$’,

$R_{n,k}$ = ‘b ‹$A_{k,1}$'› b [$A_{k,1}$] b ‹$A_{k,2}$'› b [$A_{k,2}$] ... b ‹A_{k,p_k-1}'› b [A_{k,p_k-1}] b ‹$R_{n,k+1}$› b [A_{k,p_k}] c_k-1 $\#_k$’,

$R_{1,s}$ = ‘0’.

Rule A5b (Rule A5a does not apply, separator $[A_{i,p_i}] = [1\ [A_{i+1,1}]\ 1\ [A_{i+1,2}]\ ...\ 1\ [A_{i+1,p_{i+1}}]\ c_{i+1}\ \#_{i+1}]$, which is an m-hyperseparator, where $p_{i+1} \geq 1$, $c_{i+1} \geq 2$ and $m \geq 1$):

S_i = 'b ‹$A_{i,1}$'› b [$A_{i,1}$] b ‹$A_{i,2}$'› b [$A_{i,2}$] ... b ‹A_{i,p_i-1}'› b [A_{i,p_i-1}] b ‹S_{i+1}› b [A_{i,p_i}] c_i-1 $\#_i$'.

Increment i, t_1, t_2, ... , t_m by 1; reset t_{m+1}, t_{m+2}, ... , t_x to 0 and repeat Rules A5a-c. ($i = t_1+1$.)

Rule A5c (Rules A5a-b do not apply):

S_i = 'b ‹$A_{i,1}$'› b [$A_{i,1}$] b ‹$A_{i,2}$'› b [$A_{i,2}$] ... b ‹A_{i,p_i}'› b [A_{i,p_i}] c_i-1 $\#_i$'.

Rule A6 (Rules A1-5 do not apply):

'a ‹n #› b' = 'a ‹n-1 #› b [n #] a ‹n-1 #› b [n #] ... [n #] a ‹n-1 #› b'
(with b 'a ‹n-1 #› b' strings).

Notes:

1. A, B, $A_{i,1}$, $A_{i,2}$, ... , A_{i,p_i} are strings of characters within separators.
2. $A_{i,1}$', $A_{i,2}$', ... , A_{i,p_i}' are strings of characters within angle brackets that are identical to the strings $A_{i,1}$, $A_{i,2}$, ... , A_{i,p_i} respectively except that the first entries of each have been reduced by 1. If $A_{i,j}$ (for some $1 \leq j \leq p_i$) begins with 1, $A_{i,j}$' begins with 0.
3. S_i and $R_{n,k}$ are string building functions which create strings of characters. The R functions involve nesting the same string of characters around itself n-1 times before being replaced by the string '0'.
4. #, #* and $\#_i$ are strings of characters representing the remainder of the array (can be null or empty).
5. A $/_n$ symbol is an n-hyperseparator, $/_n$ enclosed by m pairs of square brackets (with no $/_k$ symbols enclosed by fewer than k+m-n pairs of square brackets, for any k) is an (n-m)-hyperseparator (when n > m) and a normal separator (or 0-hyperseparator) otherwise. A separator containing no slashes whatsoever is a normal separator.
6. The comma is used as shorthand for the [1] separator.
7. $/_n$ is used as shorthand for the [1 $/_{n+1}$ 2] separator.

My Hierarchical Hyper-Nested Array Notation has a limit ordinal not of $\theta(\varepsilon_{\Omega+2})$, not of $\theta(\zeta_{\Omega+1})$, not of $\theta(\varphi(\omega, \Omega+1))$, not of $\theta(\theta_1(\Omega))$, not even of $\theta(\Omega_2)$ or $\theta(\varepsilon(\Omega_2+1))$ – but of $\theta(\Omega_\omega)$, the proof theoretic ordinal of the subsystem Π^1_1-CA_0 of second-order arithmetic! It has not only gone so vastly beyond the Bachmann-Howard ordinal of the previous notation, it has achieved it with a simplified Angle Bracket Rule A5 (with only three subrules instead of five). The main key to this has been the battery of t-counters when Rule A5b is repeatedly applied, tallying the various consecutive levels (or above) of the [A_{i,p_i}] hyperseparators; when this is finished by Rule A5a ([A_{i,p_i}] = $/_m$), a gigantic (m-1)-hyperseparator nesting operation begins for the topmost layers of the [A_{i,p_i}] separators, the number of topmost layers being determined by the t-counter corresponding to the mth level of hyperseparators (t_m+1). An increase in the first t-counter (t_1) from 0 to 1 gets us beyond $\Gamma_0 = \theta(\Omega)$, another increment of t_1 to 2 takes us past $\theta(\Omega$^Ω^$\Omega)$, and $t_1 = 3$ is needed to exceed $\theta(\Omega$^Ω^Ω^Ω^$\Omega)$. An increase in the second t-counter (t_2) from 0 to 1 brings us further than $\theta(\Omega_2)$, another increment of t_2 to 2 achieves $\theta(\Omega_2$^Ω_2^$\Omega_2)$, and so on. In general, $t_n = k$ is required to go beyond $\theta(\Omega_n$^^(2k-1)).

This notation works for simple nested arrays up to ε_0 level without requiring Rules A2, A5a and A5b. Rule A5a is required to reach ε_0 (represented by the [1 / 2] separator), Rule A2 is needed to achieve $\varphi(\omega, 0)$ (or use the [1 [2 $/_2$ 2] 2] separator) and Rule A5b is required to get to Γ_0 (or process [1 [1 / 2 $/_2$ 2] 2] or beyond).

An m-hyperseparator ($m \geq 1$) is either the $/_m$ symbol or a separator of the form

$[n_1\ [X_1]\ n_2\ [X_2]\ \ldots\ [X_k]\ n_{k+1}]$,

where k ≥ 1, at least one of the $[X_i]$ is an (m+1)-hyperseparator (with the $/_{m+1}$ symbol used in place of [1 $/_{m+2}$ 2]) and none of the $[X_i]$ is an (m+2)- or higher order hyperseparator. In Rule A3, the levels of two n-hyperseparators [A] and [B] are determined by the highest ranking (n+1)-hyperseparator within their ‘base layers’, then the numbers of them when they are identical. When the numbers are equal, this is repeated for the subarrays of [A] and [B] to the right of the rightmost highest ranking (n+1)-hyperseparator. When the highest ranking (n+1)-hyperseparators and their numbers within the subarrays are identical, this is repeated again for the subarrays within the subarrays, until no more (n+1)-hyperseparators remain, in which case the two subarrays become (n-1)-hyperseparators and the entire process is repeated for them as two (n-1)-hyperseparators [A] and [B]. If no more 2- or higher order hyperseparators remain, the ordinal levels of the two subarrays (normal separators when placed within square brackets) are then considered (see page 22 of Beyond Bird’s Nested Arrays I for details on how the levels of two normal separators are determined). If the ordinal levels are the same, the string of characters from the final 2- or higher hyperseparator onwards from each of [A] and [B] are deleted, and the entire process is repeated for the truncated [A] and [B], until these become normal separators (in which case their ordinal levels are taken into account). When we get a lower level or number for [A] than for [B] on some measure, then the original [A] ranks lower than the original [B] and the ‘[A] 1’ string is deleted. (The levels of (n+1)-hyperseparators are first determined by the highest ranking (n+2)-hyperseparator within their ‘base layers’, and so on.)

In Rule A5a, where I have written “separator $[A_{i,p_i}]$ = [1 $/_{m+1}$ 2] = $/_m$, where m ≥ 1”, the $/_m$ has been written as a shorthand for [1 $/_{m+1}$ 2]. I have included the [1 $/_{m+1}$ 2] since A_{i,p_i} is a string within square brackets, beginning with a number. There may exist $[A_{i,j}]$ = [1 $/_{m+1}$ 2] for some j < p_i, in which case we need to find ‘b ‹$A_{i,j}$’› b’, which would be ‘b ‹0 $/_{m+1}$ 2› b’ = ‘b’ (as m+1 ≥ 2). Some form of shorthand is necessary since there are an infinite number of ways of writing $/_n$, namely

[1 $/_{n+1}$ 2], [1 [1 $/_{n+2}$ 2] 2], [1 [1 [1 $/_{n+3}$ 2] 2] 2], ...

The following function grows so astonishing fast that its growth rate is of the magnitude of the $\theta(\Omega_\omega)$ ordinal:

U(n) = {3, n [1 [1 [... [1 [1 $/_n$ 1 $/_n$ 2] 2] ...] 2] 2] 2} (with n layers of square brackets).

The first four values of the U function are as follows:

U(1) = 3,
U(2) = {3, 2 [1 [1 $/_2$ 1 $/_2$ 2] 2] 2}
= {3 ‹0 [1 $/_2$ 1 $/_2$ 2] 2› 2}
= {3 ‹2 ‹2 $/_2$ 2› 2› 2}
= {3 ‹2 / 2 [2 $/_2$ 2] 2 / 2› 2},
U(3) = {3, 3 [1 [1 [1 $/_3$ 1 $/_3$ 2] 2] 2] 2}
= {3 ‹0 [1 [1 $/_3$ 1 $/_3$ 2] 2] 2› 3}
= {3 ‹3 ‹3 ‹3 $/_3$ 3 ‹3 $/_3$ 3› 3› 3› 3› 3},
U(4) = {3, 4 [1 [1 [1 [1 $/_4$ 1 $/_4$ 2] 2] 2] 2] 2}
= {3 ‹0 [1 [1 [1 $/_4$ 1 $/_4$ 2] 2] 2] 2› 4}
= {3 ‹4 ‹4 ‹4 ‹4 $/_4$ 4 ‹4 $/_4$ 4 ‹4 $/_4$ 4› 4› 4› 4› 4› 4› 4}.

While the rate of growth of Friedman’s TREE sequence (finite form of Kruskal’s Tree Theorem) is only just beyond the level of the small Veblen ordinal (θ(Ω^ω)), and TREE(3) is greater than

{3, 6, 3 [1 [1 / 1, 2 $/_2$ 2] 2] 2}

in my Hierarchical Hyper-Nested Array Notation, there are a few functions that grow as rapidly as the U function, which is at the $\theta(\Omega_\omega)$ level in the fast-growing hierarchy. These include the Extended

Kruskal Theorem (Kruskal's Tree Theorem extended to labelled trees, which have vertices labelled from 1 to n), the Graph Minor Theorem (or Subcubic Graph Numbers) and Buchholz Hydras (with ω labels removed).

Imagine how huge this number must be:

U(U(U(...U(3)...))) (with U(3) U's).

It surely must be scraping infinity!

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