# The Fast-Growing Hierarchy in Terms of Bird's Array Notations

Let μ be a large countable ordinal such that a fundamental sequence (a strictly increasing sequence of ordinals whose supremum is a limit ordinal) is assigned to every limit ordinal less than μ. A fast-growing hierarchy of functions $f_\alpha$, for α < μ, is then defined as follows:

$f_0(n) = n+1$,

$f_{\alpha+1}(n) = f_\alpha^n(n)$,

$f_\alpha(n) = f_{\alpha[n]}(n)$ (if α is a limit ordinal).

Here $f_\alpha^n(n) = f_\alpha(f_\alpha(...(f_\alpha(n))...))$ denotes the nth iterate of $f_\alpha$ applied to n, and α[n] denotes the nth element of the fundamental sequence assigned to the limit ordinal α. The ordinal α[n] tends to α, as n tends to ω (supremum of the finite numbers). $f_\alpha(1) = 2$ for all α < μ. The function $f_\alpha(n)$ is equivalent to $h_{\omega\char`^\alpha}(n)$ in the Hardy hierarchy of functions.

Some examples of fundamental sequences are shown below. For limit ordinals λ, written in Cantor normal form:

If λ = ω, then λ[n] = n;

if $\lambda = \omega\char`^\alpha_1 + \omega\char`^\alpha_2 + ... + \omega\char`^\alpha_k$ for $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_k$,

then $\lambda[n] = \omega\char`^\alpha_1 + \omega\char`^\alpha_2 + ... + \omega\char`^\alpha_{k-1} + (\omega\char`^\alpha_k)[n]$;

if λ = ω^(α+1), then λ[n] = (ω^α)n;

if λ = ω^α for a limit ordinal α, then λ[n] = ω^(α[n]);

if $\lambda = \varepsilon_0$, then λ[0] = 0 and λ[n+1] = ω^λ[n], which is the same as λ[n+1] = ω^^n;

if λ = φ(α+1, 0), then λ[0] = 0 and λ[n+1] = φ(α, λ[n]);

if λ = φ(α+1, β+1), then λ[0] = φ(α+1, β) + 1 and λ[n+1] = φ(α, λ[n]);

if λ = φ(α, β) for a limit ordinal β < λ, then λ[n] = φ(α, β[n]);

if λ = φ(α, 0) for a limit ordinal α < λ, then λ[n] = φ(α[n], 0);

if λ = φ(α, β+1) for a limit ordinal α, then λ[n] = φ(α[n], φ(α, β) + 1);

if $\lambda = \Gamma_0$, then λ[0] = 0 and λ[n+1] = φ(λ[n], 0).

The φ function (Veblen function) shown is the binary (2-argument) version. The extended φ function (with any finite number of arguments) can have a string of one or more 0's (or no arguments) in between the two arguments of the binary φ function shown in each of the λ formulae, and an arbitrary string of ordinal variables $\alpha_1, \alpha_2, ... , \alpha_k$ prior to the first of the two arguments of the binary φ function – these values remain unchanged in the λ[n] formulae.

The two-argument ordinal collapsing θ function used in the Beyond Bird's Nested Arrays documents relates to the extended φ function as follows:

$\theta((\Omega\char`^k)\alpha_k + ... + (\Omega\char`^2)\alpha_2 + \Omega\alpha_1 + \alpha_0, \beta) = \varphi(\alpha_k, ... , \alpha_2, \alpha_1, \alpha_0, \beta)$,

where Ω denotes the smallest uncountable ordinal. The φ function can be extended further into transfinite arguments. When β = 0, the θ function can be written as a single argument function; for instance, θ(α) = θ(α, 0). The significance of Ω is that

θ(α+Ω) is the limit of θ(α+θ(α+θ(α+... ,

θ(αΩ) is the limit of θ(αθ(αθ(α... ,

θ(α^Ω) is the limit of θ(α^θ(α^θ(α^... ,

$\theta(\alpha_1\char`^\alpha_2\char`^...\char`^\alpha_k\char`^\Omega)$ is the limit of $\theta(\alpha_1\char`^\alpha_2\char`^...\char`^\alpha_k\char`^\theta(\alpha_1\char`^\alpha_2\char`^...\char`^\alpha_k\char`^\theta(\alpha_1\char`^\alpha_2\char`^...\char`^\alpha_k\char`^...$ .

While $\theta(\Omega) = \Gamma_0$ is the Feferman-Schütte ordinal, θ(Ω^ω) is the small Veblen ordinal (limit of φ function with finite arguments) and θ(Ω^Ω) is the large Veblen ordinal (an ordinal so large that the φ function would require as many arguments as the ordinal itself – in other words, the absolute limit of the φ function).

It can be shown that, for n ≥ 1 and k < ω,

$f_1(n)$ = 2n,
$f_2(n)$ = (2^n)n ≥ 2^n,
$f_3(n)$ ≥ 2^2^2^...^2^n (with n 2's)
≥ 2^^n (using Knuth's Up-arrow Notation),
$f_4(n)$ ≥ 2^^2^^...^^2^^n (with n 2's)
≥ 2^^^n,
$f_k(n)$ ≥ 2^^^···^n (with k-1 up-arrows)
= {2, n, k-1} (using Bird's Linear Array Notation),
$f_\omega(n) = f_n(n)$
≥ {2, n, n-1}.

Note that {a, b, 0} = ab, since Bird's Linear Array Notation with 3 entries can be extended downwards by setting the third entry to 0. Exponentiation (single arrow) is repeated multiplication (no arrows) just as k arrows is repeated k-1 arrows. When n = 1, $f_\omega(1) = f_1(1)$ = {2, n, n-1} = {2, 1, 0} = 2×1 = 2.

The function $f_k(n)$ is primitive recursive for all k < ω. The $f_\omega(n)$ function diagonalises over the $f_k(n)$ functions (for finite k) and is the lowest function in the fast-growing hierarchy that is not primitive recursive. The single-argument Ackermann function grows as rapidly as $f_\omega(n)$.

For n ≥ 2,

$f_\omega(n)$ = {2, n, n-1}
= {2, {2, n-1, n-1}, n-2}
> {n, n, n-2} + 2.

This holds for n ≥ 4, since {2, 4, 3} = 2^^65,536 > 4^4^4^4 + 2 = {4, 4, 2} + 2 when n = 4. For higher values of n, the second entry of the 3-entry array ending in n-2 (height of (n-2)-arrow tower) is much more important than the first entry (number that is copied and placed at every storey of the tower). This is because, for each value of n ≥ 5, the arrays {2, {2, n-1, n-1}, n-2} = 2^^X and {n, n, n-2} = n^^Y, where X and Y are large integers, and since n < {2, n-1, n-1} < X, when 2^^X is written as the highest power tower of 2's with a number equal to or greater than n on top, its height (call it X') is still far greater than Y (height of power tower of n's in n^^Y) – when evaluating power towers with the same numbers on top the heights are far more important than the numbers in the storeys (below the top). In fact, X' is so much greater than Y that any comparisons between 2 and n pale into insignificance.

The above result also holds for n = 2 and n = 3, since

$f_\omega(2) = f_2(2)$ = 8
> {n, n, n-2} + 2 = {2, 2, 0} + 2 = 2×2 + 2 = 6,
$f_\omega(3) = f_3(3)$ ≥ 2^2^2^3 = 2^256
> {n, n, n-2} + 2 = 29.

For n ≥ 2,

$f_{\omega+1}(n)$ > {n, n, 1, 2}
= {n, n, {n, n, { ... {n, n, n}...}}} (with n-1 pairs of curly brackets),

as $f_{\omega+1}(n) = f_\omega^{n}(n)$
> $f_\omega^{n-1}(n+2)$
> $f_\omega^{n-2}$({n, n, (n+2)-2} + 2) = $f_\omega^{n-2}$({n, n, n} + 2)
> $f_\omega^{n-3}$({n, n, ({n, n, n}+2)-2} + 2) = $f_\omega^{n-3}$({n, n, {n, n, n}} + 2)
> {n, n, {n, n, { ... {n, n, n}...}}} (with n-1 pairs of curly brackets).

This also holds for n = 1, since $f_{\omega+1}(1)$ = 2 > {1, 1, 1, 2} = 1.

$f_{\omega+1}(64)$ is greater than Graham's Number since, for $n \geq 5$,

$f_{\omega}(n) > \{3, 3, n\text{-}2\} + 2$,

which means that,

$$
\begin{aligned}
f_{\omega+1}(64) &> f_{\omega}^{64}(6) \\
&> f_{\omega}^{63}(\{3, 3, 4\} + 2) \\
&> f_{\omega}^{62}(\{3, 3, \{3, 3, 4\}\} + 2) \\
&> f_{\omega}^{61}(\{3, 3, \{3, 3, \{3, 3, 4\}\}\} + 2) \\
&> \{3, 3, \{3, 3, \{ \ldots \{3, 3, 4\} \ldots\}\}\} \quad \text{(with 64 pairs of curly brackets, Graham's Number).}
\end{aligned}
$$

It can be shown that, for $n \geq 1$ and $k < \omega$,

$f_{\omega+2}(n) > \{n, n, 2, 2\}$,

$f_{\omega+3}(n) > \{n, n, 3, 2\}$,

$f_{\omega+k}(n) > \{n, n, k, 2\}$,

$f_{\omega 2}(n) = f_{\omega+n}(n) > \{n, n, n, 2\}$,

$f_{\omega 2+1}(n) > \{n, n, 1, 3\}$,

$f_{\omega 2+k}(n) > \{n, n, k, 3\}$,

$f_{\omega 3}(n) > \{n, n, n, 3\}$,

$f_{\omega k}(n) > \{n, n, n, k\}$,

$f_{\omega\text{^}2}(n) = f_{\omega n}(n) > \{n, n, n, n\}$

$\geq n \to n \to \ldots \to n$ (with n entries, Conway's Chained Arrow Notation),

$f_{\omega\text{^}2 + 1}(n) > \{n, n, 1, 1, 2\}$,

$f_{\omega\text{^}2 + k}(n) > \{n, n, k, 1, 2\}$,

$f_{\omega\text{^}2 + \omega}(n) > \{n, n, n, 1, 2\}$,

$f_{\omega\text{^}2 + \omega + k}(n) > \{n, n, k, 2, 2\}$,

$f_{\omega\text{^}2 + \omega k}(n) > \{n, n, n, k, 2\}$,

$f_{(\omega\text{^}2)2}(n) > \{n, n, n, n, 2\}$,

$f_{(\omega\text{^}2)k}(n) > \{n, n, n, n, k\}$,

$f_{\omega\text{^}3}(n) > \{n, n, n, n, n\}$,

$f_{\omega\text{^}k}(n) > \{n, n, n, \ldots , n\}$ (with k+2 n's).

Using Bird's Multi-Dimensional Array Notation (in which a number in square brackets denotes a separator, a comma being a shorthand for [1] in this notation),

$f_{\omega\text{^}\omega}(n) > \{n, n+2\ [2]\ 2\}$

$= \{n, n, n, \ldots , n\}$ (with n+2 n's),

$f_{\omega\text{^}\omega + 1}(n) > \{n, n, 2\ [2]\ 2\}$,

$f_{\omega\text{^}\omega + k}(n) > \{n, n, k+1\ [2]\ 2\}$,

$f_{\omega\text{^}\omega + \omega}(n) > \{n, n, n+1\ [2]\ 2\}$,

$f_{\omega\text{^}\omega + \omega + 1}(n) > \{n, n, 1, 2\ [2]\ 2\}$,

$f_{\omega\text{^}\omega + \omega + k}(n) > \{n, n, k, 2\ [2]\ 2\}$,

$f_{\omega\text{^}\omega + \omega k}(n) > \{n, n, n, k\ [2]\ 2\}$,

$f_{\omega\text{^}\omega + \omega\text{^}2}(n) > \{n, n, n, n\ [2]\ 2\}$,

$f_{\omega\text{^}\omega + \omega\text{^}k}(n) > \{n, n, n, \ldots , n\ [2]\ 2\}$ (with k+2 n's),

$f_{(\omega\text{^}\omega)2}(n) > \{n, n+2\ [2]\ 3\}$

$= \{n, n, n, \ldots , n\ [2]\ 2\}$ (with n+2 n's),

$f_{(\omega\text{^}\omega)k}(n) > \{n, n+2\ [2]\ k+1\}$

$= \{n, n, n, \ldots , n\ [2]\ k\}$ (with n+2 n's),

$f_{\omega\text{^}(\omega+1)}(n) > \{n, n, n, \ldots , n\ [2]\ n\}$ (with n+2 n's before [2]),

$f_{\omega\text{^}(\omega+1) + 1}(n) > \{n, n\ [2]\ 1, 2\}$,

$f_{\omega\text{^}(\omega+1) + k}(n) > \{n, n, k\ [2]\ 1, 2\}$,

$f_{\omega\text{^}(\omega+1)+\omega}(n)$ > {n, n, n [2] 1, 2},
$f_{\omega\text{^}(\omega+1)+\omega\text{^}k}(n)$ > {n, n, ... , n [2] 1, 2} (with k+2 n's),
$f_{\omega\text{^}(\omega+1)+\omega\text{^}\omega}(n)$ > {n, n, ... , n [2] 1, 2} (with n+2 n's),
$f_{\omega\text{^}(\omega+1)+(\omega\text{^}\omega)2}(n)$ > {n, n, ... , n [2] 2, 2} (with n+2 n's),
$f_{\omega\text{^}(\omega+1)+(\omega\text{^}\omega)k}(n)$ > {n, n, ... , n [2] k, 2} (with n+2 n's),
$f_{(\omega\text{^}(\omega+1))2}(n)$ > {n, n, ... , n [2] n, 2} (with n+2 n's before [2]),
$f_{(\omega\text{^}(\omega+1))k}(n)$ > {n, n, ... , n [2] n, k} (with n+2 n's before [2]),
$f_{\omega\text{^}(\omega+2)}(n)$ > {n, n, ... , n [2] n, n} (with n+2 n's before [2]),
$f_{\omega\text{^}(\omega+k)}(n)$ > {n, n, ... , n [2] n, n, ... , n} (with n+2 n's before [2] and k n's after [2]),
$f_{\omega\text{^}(\omega2)}(n)$ > {n, n, ... , n [2] n, n, ... , n} (with n+2 n's before [2] and n n's after [2]),
$f_{\omega\text{^}(\omega k)}(n)$ > {n,n,..,n [2] n,n,..,n [2] ... [2] n,n,..,n} (with k 'rows', each containing n n's),
$f_{\omega\text{^}\omega\text{^}2}(n)$ > {n, n [3] 2}
= {n,n,..,n [2] n,n,..,n [2] ... [2] n,n,..,n} (2 dimensional n^2 array of n's),
$f_{\omega\text{^}\omega\text{^}3}(n)$ > {n, n [4] 2}
= {n,n,..,n [2] n,n,..,n [2] ... [2] n,n,..,n [3]
n,n,..,n [2] n,n,..,n [2] ... [2] n,n,..,n [3]
..................................................... [3]
n,n,..,n [2] n,n,..,n [2] ... [2] n,n,..,n} (3 dimensional n^3 array of n's),
$f_{\omega\text{^}\omega\text{^}k}(n)$ > {n, n [k+1] 2} (k dimensional n^k array of n's).

Using Bird's Hyper-Dimensional Array Notation (in which the separators themselves become arrays),
$f_{\omega\text{^}\omega\text{^}\omega}(n)$ > {n, n [1, 2] 2}
= {n, n [n+1] 2} (n dimensional n^n array of n's),
$f_{\omega\text{^}\omega\text{^}(\omega+k)}(n)$ > {n, n [k+1, 2] 2},
$f_{\omega\text{^}\omega\text{^}(\omega2)}(n)$ > {n, n [1, 3] 2},
$f_{\omega\text{^}\omega\text{^}(\omega k)}(n)$ > {n, n [1, k+1] 2},
$f_{\omega\text{^}\omega\text{^}\omega\text{^}2}(n)$ > {n, n [1, 1, 2] 2},
$f_{\omega\text{^}\omega\text{^}\omega\text{^}k}(n)$ > {n, n [1, 1, ... , 1, 2] 2} (with k 1's).

Using Bird's Nested Array Notation (in which separator arrays can be nested inside themselves),
$f_{\omega\text{^}\omega\text{^}\omega\text{^}\omega}(n)$ > {n, n [1 [2] 2] 2},
$f_{\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}2}(n)$ > {n, n [1 [3] 2] 2},
$f_{\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}k}(n)$ > {n, n [1 [k+1] 2] 2},
$f_{\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}\omega}(n)$ > {n, n [1 [1, 2] 2] 2},
$f_{\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}2}(n)$ > {n, n [1 [1, 1, 2] 2] 2},
$f_{\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}\omega\text{^}k}(n)$ > {n, n [1 [1, 1, ... , 1, 2] 2] 2} (with k 1's),
$f_{\omega\text{^^}6}(n)$ > {n, n [1 [1 [2] 2] 2] 2},
$f_{\omega\text{^^}7}(n)$ > {n, n [1 [1 [1, 2] 2] 2] 2},
$f_{\omega\text{^^}8}(n)$ > {n, n [1 [1 [1 [2] 2] 2] 2] 2}.

In general, for 2 ≤ k < ω,
$f_{\omega\text{^^}k}(n)$ > {n, n [1 [1 [ ... [1 [1, 2] 2] ... ] 2] 2] 2} (with (k-1)/2 pairs of square brackets, k odd)
> {n, n [1 [1 [ ... [1 [2] 2] ... ] 2] 2] 2} (with k/2 pairs of square brackets, k even).
This has limit ordinal $\varepsilon_0$ in the fast-growing hierarchy.

Each new entry in the array adds one to the corresponding power of ω in the f subscript, each new row adds ω to the power of ω, each new plane adds ω^2 to the power of ω, and so on. Each new dimension adds one to the corresponding power of ω^ω, each new entry in a separator array adds one to the power of ω^ω^ω, each new dimension within it adds one to the power of ω^ω^ω^ω, each new entry in a separator array within a separator array adds one to the power of ω^ω^ω^ω^ω, and so on. Turning a single-entry separator (within various levels or nests of separator arrays, if any) into a linear array (separated by commas or [1]'s) adds one to the corresponding height of the exponential tower of ω's; turning it into a multi-dimensional array (separated by [2]'s or higher) adds two to the height of the ω power tower.

The limit ordinals of Knuth's Up-arrow Notation, Conway's Chained Arrow Notation and Bird's Array Notations thus far, in the fast-growing hierarchy, are shown below:

| **Notation** | **Limit ordinal** | **Fast-growing functions at limit ordinal** |
|---|---|---|
| Knuth's Up-arrows | ω | Ackermann Function |
| Conway's Chained Arrows | ω^2 | |
| Bird's Linear Arrays | ω^ω | Friedman's n(k) (Block Subsequence Theorem) |
| Bird's Multi-Dimensional Arrays | ω^ω^ω | |
| Bird's Hyper-Dimensional Arrays | ω^ω^ω^ω | |
| Bird's Nested Arrays | $\varepsilon_0$ | Goodstein Sequence, Fusible Numbers |

We now proceed into Bird's Hyper-Nested Arrays (see Beyond Bird's Nested Arrays I and II) and Nested Hyper-Nested Arrays (see Beyond Bird's Nested Arrays III). Since $f_{\omega\text{^^}(2n-1)}(n)$ > {n, n [1 \ 2] 2} grows twice as fast as $f_{\varepsilon_0}(n) = f_{\omega\text{^^}(n-1)}(n)$, and the sequence {n, n [1 \ k+3] 2}, for some k, grows somewhat more quickly than $f_{\varepsilon_{k+1}}(n) = f_{\omega\text{^}\omega\text{^}\ldots\text{^}\omega\text{^}(\varepsilon_k+1)}(n)$ (with n ω's), it can be shown that

$f_{\varepsilon_0 + 1}(n)$ > {n, n [1 \ 2] 2},
$f_{\varepsilon_k + 1}(n)$ > {n, n [1 \ k+2] 2},
$f_{\varepsilon_\omega}(n)$ > {n, n [1 \ 1, 2] 2},
$f_{\varepsilon(\varepsilon_0) + 1}(n)$ > {n, n [1 \ 1 [1 \ 2] 2] 2},
$f_{\zeta_0}(n) = f_{\varphi(2, 0)}(n)$ > {n, n [1 \ 1 \ 2] 2},
$f_{\varphi(3, 0)}(n)$ > {n, n [1 \ 1 \ 1 \ 2] 2},
$f_{\varphi(k, 0)}(n)$ > {n, n [1 \ 1 \ ... \ 1 \ 2] 2} (with k 1's),
$f_{\varphi(\omega, 0)}(n)$ > {n, n [1 [2 ¬ 2] 2] 2} (\ is shorthand for [1 ¬ 2]),
$f_{\varphi(\omega\text{^}k, 0)}(n)$ > {n, n [1 [k+1 ¬ 2] 2] 2},
$f_{\varphi(\omega\text{^}\omega, 0)}(n)$ > {n, n [1 [1, 2 ¬ 2] 2] 2},
$f_{\varphi(\omega\text{^}\omega\text{^}\omega, 0)}(n)$ > {n, n [1 [1 [2] 2 ¬ 2] 2] 2},
$f_{\varphi(\omega\text{^}\omega\text{^}\omega\text{^}\omega, 0)}(n)$ > {n, n [1 [1 [1, 2] 2 ¬ 2] 2] 2},
$f_{\varphi(\varepsilon_0, 0) + 1}(n)$ > {n, n [1 [1 \ 2 ¬ 2] 2] 2},
$f_{\varphi(\varphi(\omega, 0), 0)}(n)$ > {n, n [1 [1 [2 ¬ 2] 2 ¬ 2] 2] 2},
$f_{\varphi(\varphi(\varepsilon_0, 0), 0) + 1}(n)$ > {n, n [1 [1 [1 \ 2 ¬ 2] 2 ¬ 2] 2] 2},
$f_{\Gamma_0}(n) = f_{\theta(\Omega)}(n)$ > {n, n [1 [1 ¬ 3] 2] 2} (θ is an ordinal collapsing function),
$f_{\varphi(1, 0, 0, 0)}(n) = f_{\theta(\Omega\text{^}2)}(n)$ > {n, n [1 [1 ¬ 4] 2] 2},
$f_{\theta(\Omega\text{^}k)}(n)$ > {n, n [1 [1 ¬ k+2] 2] 2},
$f_{\theta(\Omega\text{^}\omega)}(n)$ > {n, n [1 [1 ¬ 1, 2] 2] 2} (growth rate small Veblen ordinal),
$f_{\theta(\Omega\text{^}\Omega)}(n)$ > {n, n [1 [1 ¬ 1 ¬ 2] 2] 2} (growth rate large Veblen ordinal),
$f_{\theta(\Omega\text{^}\Omega\text{^}\omega)}(n)$ > {n, n [1 [1 [2 $\backslash_3$ 2] 2] 2] 2} (¬ or $\backslash_2$ is shorthand for [1 $\backslash_3$ 2]),
$f_{\theta(\Omega\text{^}\Omega\text{^}\Omega)}(n)$ > {n, n [1 [1 [1 $\backslash_3$ 3] 2] 2] 2},
$f_{\theta(\Omega\text{^}\Omega\text{^}\Omega\text{^}\Omega)}(n)$ > {n, n [1 [1 [1 $\backslash_3$ 1 $\backslash_3$ 2] 2] 2] 2},
$f_{\theta(\Omega\text{^}\Omega\text{^}\Omega\text{^}\Omega\text{^}\Omega)}(n)$ > {n, n [1 [1 [1 [1 $\backslash_4$ 3] 2] 2] 2] 2} ($\backslash_k$ is shorthand for [1 $\backslash_{k+1}$ 2]).

The rate of growth of Friedman's TREE sequence (finite form of Kruskal's Tree Theorem) exceeds that of $f_{\Gamma_0}(n)$. There is a less rigorous proof that the growth rate exceeds that of $f_{\theta(\Omega\text{^}\omega)}(n)$. See page 10 of Beyond Bird's Nested Arrays II for more details on this function.

Separators of the form $[X_1 \backslash X_2 \backslash \ldots \backslash X_k]$ (where the $X_i$ are strings of characters and the backslash is a hyperseparator marking successive levels of the binary φ function, represented by the left-hand argument) provide the foundations of Bird's Hyper-Nested Arrays – $X_1$ corresponds to ordinary addition, multiplication and exponentiation (or the φ(0, α) function); $X_2$ corresponds to the epsilon numbers (φ(1, α)) and $X_k$ corresponds to ordinals of the form φ(k-1, α). For example, the separator

$$[n_1+1 \backslash n_2+1 \backslash \ldots \backslash n_{k-1}+1 \backslash n_k+2]$$

where $n_1, n_2, n_3, \ldots, n_k$ are non-negative integers, corresponds to the ordinal

$$\varepsilon(\zeta(\varphi(3, \varphi( \ldots \varphi(k-2, \varphi(k-1, n_k) + n_{k-1}) \ldots ) + n_4) + n_3) + n_2) \wedge \omega \wedge n_1$$

in the fast-growing hierarchy, i.e. when α is the above ordinal,

$$f_\alpha(n) > \{n, n\ [n_1+1 \backslash n_2+1 \backslash \ldots \backslash n_{k-1}+1 \backslash n_k+2]\ 2\}.$$

The [2 ¬ 2] hyperseparator (backslash is [1 ¬ 2] while ¬ is a 2-hyperseparator) gets us beyond φ(ω, 0) and into the second dimension of 'hyperspace', while nested levels of [ ¬ 2] inside [ ¬ 2] enable us to reach $\Gamma_0$ and the [1 ¬ 3] hyperseparator. Nesting on the right-hand side of the ¬ sign leads us to the large Veblen ordinal and ¬ chains. Then by analogy of ¬ with \ we repeat this process to create higher order hyperseparators $\backslash_3$, $\backslash_4$, $\backslash_5$ and so on, where $\backslash_k$ is a k-hyperseparator and shorthand for $[1 \backslash_{k+1} 2]$. The limit ordinal of all of this is $\theta(\varepsilon_{\Omega+1})$, the Bachmann-Howard ordinal.

When X is a string of characters such that [X] is a separator with level α (separates α dimensional spaces – containing up to ω^α arguments – within an array, e.g. comma or [1] has level 0, [n+1] has level n, [1, 2] has level ω, [1 \ 2] has level $\varepsilon_0$), then

$$f_{\omega\wedge\omega\wedge\alpha + 1}(n) > \{n, n\ [X]\ 2\}.$$

In fact,

$$f_{\omega\wedge\omega\wedge\alpha}(n) > \{n, n\ [X]\ 2\}$$

holds true in virtually all cases. It holds true for all $0 \le \alpha < \varepsilon_0$, but not when α is $\varepsilon_0$ or an epsilon number of the form $\varepsilon_{\beta+1}$, $\varepsilon(\varepsilon(\varepsilon(\ldots(\varepsilon_0)\ldots)))$ or $\varepsilon(\varepsilon(\varepsilon(\ldots(\varepsilon_{\beta+1})\ldots)))$, where there are a finite number of ε's in the expression.

It can be shown that, for n ≥ 1, k < ω and [X] being larger than all of the other separators in the array,

$f_{\omega\wedge\omega\wedge\alpha + 2}(n) > \{n, n, 2\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + k}(n) > \{n, n, k\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \omega}(n) > \{n, n, n\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \omega + k}(n) > \{n, n, k, 2\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \omega k}(n) > \{n, n, n, k\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \omega\wedge 2}(n) > \{n, n, n, n\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \omega\wedge k}(n) > \{n, n, n, \ldots, n\ [X]\ 2\}$ (with k+2 n's),
$f_{\omega\wedge\omega\wedge\alpha + \omega\wedge\omega}(n) > \{n, n+2\ [2]\ 2\ [X]\ 2\}$
$= \{n, n, n, \ldots, n\ [X]\ 2\}$ (with n+2 n's),
$f_{\omega\wedge\omega\wedge\alpha + \omega\wedge\omega\wedge k}(n) > \{n, n\ [k+1]\ 2\ [X]\ 2\}$ (k dimensional n^k array of n's before [X]),
$f_{\omega\wedge\omega\wedge\alpha + \omega\wedge\omega\wedge\omega}(n) > \{n, n\ [1, 2]\ 2\ [X]\ 2\}$
$= \{n, n\ [n+1]\ 2\ [X]\ 2\}$ (n dimensional n^n array of n's before [X]),
$f_{\omega\wedge\omega\wedge\alpha + \omega\wedge\omega\wedge\omega\wedge\omega}(n) > \{n, n\ [1\ [2]\ 2]\ 2\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \omega\wedge\omega\wedge\omega\wedge\omega\wedge\omega}(n) > \{n, n\ [1\ [1, 2]\ 2]\ 2\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \varepsilon_0 + 1}(n) > \{n, n\ [1 \backslash 2]\ 2\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \varphi(\omega, 0)}(n) > \{n, n\ [1\ [2 \neg 2]\ 2]\ 2\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \Gamma_0}(n) > \{n, n\ [1\ [1 \neg 3]\ 2]\ 2\ [X]\ 2\}$,
$f_{\omega\wedge\omega\wedge\alpha + \theta(\Omega\wedge\Omega)}(n) > \{n, n\ [1\ [1 \neg 1 \neg 2]\ 2]\ 2\ [X]\ 2\}$,
$f_{(\omega\wedge\omega\wedge\alpha)2 + 1}(n) > \{n, n\ [X]\ 3\}$,
$f_{(\omega\wedge\omega\wedge\alpha)k + 1}(n) > \{n, n\ [X]\ k+1\}$,

$f_{\omega\wedge(\omega\wedge\alpha + 1)}(n)$ > {n, n [X] n+1},
$f_{\omega\wedge(\omega\wedge\alpha + 1) + 1}(n)$ > {n, n [X] 1, 2},
$f_{\omega\wedge(\omega\wedge\alpha + 1) + (\omega\wedge\omega\wedge\alpha)k + 1}(n)$ > {n, n [X] k+1, 2},
$f_{(\omega\wedge(\omega\wedge\alpha + 1))2}(n)$ > {n, n [X] n+1, 2},
$f_{(\omega\wedge(\omega\wedge\alpha + 1))k}(n)$ > {n, n [X] n+1, k},
$f_{\omega\wedge(\omega\wedge\alpha + 2)}(n)$ > {n, n [X] n+1, n},
$f_{\omega\wedge(\omega\wedge\alpha + 2) + 1}(n)$ > {n, n [X] 1, 1, 2},
$f_{\omega\wedge(\omega\wedge\alpha + k) + 1}(n)$ > {n, n [X] 1, 1, ... , 1, 2} (with k 1's),
$f_{\omega\wedge(\omega\wedge\alpha + \omega)}(n)$ > {n, n [X] 1 [2] 2},
$f_{\omega\wedge(\omega\wedge\alpha + \omega\wedge k)}(n)$ > {n, n [X] 1 [k+1] 2},
$f_{\omega\wedge(\omega\wedge\alpha + \omega\wedge\omega)}(n)$ > {n, n [X] 1 [1, 2] 2}
= {n, n [X] 1 [n+1] 2},
$f_{\omega\wedge(\omega\wedge\alpha + \omega\wedge\omega\wedge\omega)}(n)$ > {n, n [X] 1 [1 [2] 2] 2},
$f_{\omega\wedge(\omega\wedge\alpha + \omega\wedge\omega\wedge\omega\wedge\omega)}(n)$ > {n, n [X] 1 [1 [1, 2] 2] 2},
$f_{(\omega\wedge\omega\wedge\alpha)\varepsilon_0 + 1}(n)$ > {n, n [X] 1 [1 \ 2] 2},
$f_{(\omega\wedge\omega\wedge\alpha)\varphi(\omega, 0)}(n)$ > {n, n [X] 1 [1 [2 ¬ 2] 2] 2},
$f_{(\omega\wedge\omega\wedge\alpha)\Gamma_0}(n)$ > {n, n [X] 1 [1 [1 ¬ 3] 2] 2},
$f_{(\omega\wedge\omega\wedge\alpha)\theta(\Omega\wedge\Omega)}(n)$ > {n, n [X] 1 [1 [1 ¬ 1 ¬ 2] 2] 2},
$f_{\omega\wedge((\omega\wedge\alpha)2) + 1}(n)$ > {n, n [X] 1 [X] 2},
$f_{\omega\wedge((\omega\wedge\alpha)k) + 1}(n)$ > {n, n [X] 1 [X] 1 ... [X] 1 [X] 2} (with k [X]'s),
$f_{\omega\wedge\omega\wedge(\alpha+1)}(n)$ > {n, n [X'] 2}
(where [X'] is identical to [X] except that the first entry is increased by 1).

When α = $\theta(\varepsilon_{\Omega+1})$ (the Bachmann-Howard ordinal), we can take λ = $\varepsilon_{\Omega+1}$ and build the fundamental sequence as follows:

α[n] = θ(λ[n]), where λ[0] = 0 and λ[n+1] = Ω^λ[n].

The first few ordinals of α[n] are:

α[0] = θ(0) = 1,
α[1] = θ(1) = $\varepsilon_0$,
α[2] = θ(Ω) = $\Gamma_0$,
α[3] = θ(Ω^Ω).

Since $f_\alpha(n) = f_{\alpha[n]}(n)$,

$f_{\varepsilon_0}(1)$ = 2,
$f_{\Gamma_0}(2)$ > {2, 2 [1 [1 ¬ 3] 2] 2} = 4 (as array reduces to {2, 2} = 2^2 = 4),
$f_{\theta(\Omega\wedge\Omega)}(3)$ > {3, 3 [1 [1 ¬ 1 ¬ 2] 2] 2},
$f_{\theta(\Omega\wedge\Omega\wedge\Omega)}(4)$ > {4, 4 [1 [1 [1 $\backslash_3$ 3] 2] 2] 2},
$f_{\theta(\Omega\wedge\Omega\wedge\Omega\wedge\Omega)}(5)$ > {5, 5 [1 [1 [1 $\backslash_3$ 1 $\backslash_3$ 2] 2] 2] 2},
$f_{\theta(\Omega\wedge\Omega\wedge\Omega\wedge\Omega\wedge\Omega)}(6)$ > {6, 6 [1 [1 [1 [1 $\backslash_4$ 3] 2] 2] 2] 2}.

In general,

$f_\alpha(n)$ > {n, n [1 [1 [ ... [1 [1 $\backslash_{(n+1)/2}$ 1 $\backslash_{(n+1)/2}$ 2] 2] ... ] 2] 2] 2}
(with (n+1)/2 pairs of square brackets, n odd)
> {n, n [1 [1 [ ... [1 [1 $\backslash_{(n+2)/2}$ 3] 2] ... ] 2] 2] 2}
(with (n+2)/2 pairs of square brackets, n even).

The H(n) function at the end of Beyond Bird's Nested Arrays III grows twice as rapidly as $f_\alpha(n)$. However H(n) grows more slowly than the function $f_{\alpha+1}(n)$ when α is the Bachmann-Howard ordinal.

Far greater ordinals exist beyond the Bachmann-Howard ordinal. Some examples are shown below (the dots denote an infinite sequence):

$\theta(\varepsilon_{\Omega+1}\wedge\Omega) = \theta(\varepsilon_{\Omega+1}\wedge\theta(\varepsilon_{\Omega+1}\wedge\theta(\varepsilon_{\Omega+1}\wedge...)))$,
$\theta(\varepsilon_{\Omega+1}\wedge\varepsilon_{\Omega+1}) = \theta(\varepsilon_{\Omega+1}\wedge\Omega\wedge\Omega\wedge\Omega\wedge...)$,
$\theta(\varepsilon_{\Omega+2}) = \theta(\varepsilon_{\Omega+1}\wedge\varepsilon_{\Omega+1}\wedge\varepsilon_{\Omega+1}\wedge...)$,
$\theta(\varepsilon_{\Omega2}) = \theta(\varepsilon(\Omega+\theta(\varepsilon(\Omega+\theta(\varepsilon(\Omega+...))))))$,
$\theta(\zeta_{\Omega+1}) = \theta(\varphi(2, \Omega+1)) = \theta(\varepsilon(\Omega+\varepsilon(\Omega+\varepsilon(\Omega+...))))$,
$\theta(\varphi(\Omega, 1)) = \theta(\varphi(\theta(\varphi(\theta(\varphi(...\theta(\varphi(\theta(\varepsilon_{\Omega+1}), \Omega+1)), ... , \Omega+1)), \Omega+1)), \Omega+1))$,
$\theta(\Omega_2) = \theta(\Gamma_{\Omega+1}) = \theta(\varphi(\varphi(\varphi(...\varphi(\varphi(\Omega, 1), 0), ... , 0), 0), 0))$,
$\theta(\varepsilon_{\Omega_2+1}) = \theta(\Omega_2\wedge\Omega_2\wedge\Omega_2\wedge...)$,
$\theta(\Omega_3) = \theta(\Gamma_{\Omega_2+1}) = \theta(\varphi(\varphi(\varphi(...\varphi(\varphi(\Omega_2, 1), 0), ... , 0), 0), 0))$,
$\theta(\Omega_\omega)$,
$\theta(\Omega_\Omega) = \theta(\Omega_{\theta(\Omega_{\theta(\Omega...)})})$,
$\theta(\Omega_{\Omega_{\Omega...}})$.

The ordinal $\Omega_\alpha$ is the $\alpha$th uncountable ordinal in the sequence starting from $\Omega_1 = \Omega$, and may be thought of as a cardinal number (similar to the Aleph numbers). (The countable ordinal $\Omega_0 = \omega$.) The ordinal $\theta(\Omega_{\Omega_{\Omega...}})$ is by no means the largest ordinal ever written down, as it only represents the first fixed point of $\alpha = \Omega_\alpha$ within the $\theta$ function, and we can create a whole new ‘superfunction’ of cardinal numbers analogous to the $\theta$ function for ordinals in order to create even more hypergigantic ordinals!

Bird’s Hierarchical Hyper-Nested Array Notation (see Beyond Bird’s Nested Arrays IV) looks much the same as Nested Hyper-Nested Arrays (in the previous document) except that it is made much more powerful, with mixing of higher order hyperseparators of various levels on the same square bracket layer allowed as well as modifications made to Angle Bracket Rule A5. The various levels of the hyperseparators are arranged in a strict hierarchy – not only does an (n+1)-hyperseparator outrank any n-hyperseparator but (n+1)-hyperseparators are only used when n-hyperseparators have been completely exhausted (the lowest level of separators are normal separators or 0-hyperseparators). This notation uses forward slashes instead of backslashes (e.g. $\backslash_n$ becomes $/_n$), in order to avoid confusion with the previous notation. The ~ symbol can be written in place of $/_2$ if this makes separator expressions easier to read. Examples include the following:

$f_{\varepsilon_0 + 1}(n) > \{n, n [1 / 2] 2\}$,
$f_{\varepsilon_k + 1}(n) > \{n, n [1 / k+2] 2\}$,
$f_{\varepsilon(\varepsilon_0) + 1}(n) > \{n, n [1 / 1 [1 / 2] 2] 2\}$,
$f_{\zeta_0}(n) = f_{\varphi(2, 0)}(n) > \{n, n [1 / 1 / 2] 2\}$,
$f_{\varphi(\omega, 0)}(n) > \{n, n [1 [2 /_2 2] 2] 2\}$ (/ is shorthand for $[1 /_2 2]$),
$f_{\varphi(\omega\wedge\omega, 0)}(n) > \{n, n [1 [1, 2 /_2 2] 2] 2\}$,
$f_{\varphi(\omega\wedge\omega\wedge\omega, 0)}(n) > \{n, n [1 [1 [2] 2 /_2 2] 2] 2\}$,
$f_{\varphi(\omega\wedge\omega\wedge\omega\wedge\omega, 0)}(n) > \{n, n [1 [1 [1, 2] 2 /_2 2] 2] 2\}$,
$f_{\varphi(\varepsilon_0, 0) + 1}(n) > \{n, n [1 [1 [1 / 2] 2 /_2 2] 2] 2\}$,
$f_{\varphi(\varphi(\varepsilon_0, 0), 0) + 1}(n) > \{n, n [1 [1 [1 [1 [1 / 2] 2 /_2 2] 2] 2 /_2 2] 2] 2\}$,
$f_{\Gamma_0}(n) = f_{\theta(\Omega)}(n) > \{n, n [1 [1 / 2 /_2 2] 2] 2\}$,
$f_{\varphi(1, 0, 0, 0)}(n) = f_{\theta(\Omega\wedge 2)}(n) > \{n, n [1 [1 / 3 /_2 2] 2] 2\}$,
$f_{\theta(\Omega\wedge\omega)}(n) > \{n, n [1 [1 / 1, 2 /_2 2] 2] 2\}$ (growth rate small Veblen ordinal),
$f_{\theta(\Omega\wedge\Omega)}(n) > \{n, n [1 [1 / 1 / 2 /_2 2] 2] 2\}$ (growth rate large Veblen ordinal),
$f_{\theta(\Omega\wedge\Omega\wedge\omega)}(n) > \{n, n [1 [1 [2 /_2 2] 2 /_2 2] 2] 2\}$,
$f_{\theta(\Omega\wedge\Omega\wedge\Omega)}(n) > \{n, n [1 [1 [1 / 2 /_2 2] 2 /_2 2] 2] 2\}$,
$f_{\theta(\Omega\wedge\Omega\wedge\Omega\wedge\Omega)}(n) > \{n, n [1 [1 [1 / 1 / 2 /_2 2] 2 /_2 2] 2] 2\}$,
$f_{\theta(\Omega\wedge\Omega\wedge\Omega\wedge\Omega\wedge\Omega)}(n) > \{n, n [1 [1 [1 [1 / 2 /_2 2] 2 /_2 2] 2 /_2 2] 2] 2\}$,

$f_{\theta(\varepsilon_{\Omega+1}) + 1}(n) > \{n, n [1 [1 /_2 3] 2] 2\}$ (growth rate Bachmann-Howard ordinal),
$f_{\theta(\varepsilon_{\Omega+1}\char94 2) + 1}(n) > \{n, n [1 [1 [1 /_2 3] 2 /_2 3] 2] 2\}$,
$f_{\theta(\varepsilon_{\Omega+1}\char94\varepsilon_{\Omega+1}) + 1}(n) > \{n, n [1 [1 [1 /_2 3] 1 [1 /_2 3] 2 /_2 3] 2] 2\}$,
$f_{\theta(\varepsilon_{\Omega+1}\char94\varepsilon_{\Omega+1}\char94\varepsilon_{\Omega+1}) + 1}(n) > \{n, n [1 [1 [1 [1 /_2 3] 2 /_2 3] 2 /_2 3] 2] 2\}$,
$f_{\theta(\varepsilon_{\Omega+2}) + 1}(n) > \{n, n [1 [1 /_2 4] 2] 2\}$,
$f_{\theta(\varepsilon_{\Omega 2})}(n) > \{n, n [1 [1 /_2 1 / 2] 2] 2\}$,
$f_{\theta(\varepsilon_{\Omega\char94 2})}(n) > \{n, n [1 [1 /_2 1 / 1 / 2] 2] 2\}$,
$f_{\theta(\varepsilon_{\Omega\char94\Omega})}(n) > \{n, n [1 [1 /_2 1 [1 / 2 /_2 2] 2] 2] 2\}$,
$f_{\theta(\varepsilon(\varepsilon_{\Omega+1})) + 1}(n) > \{n, n [1 [1 /_2 1 [1 /_2 3] 2] 2] 2\}$,
$f_{\theta(\varepsilon(\varepsilon(\varepsilon_{\Omega+1}))) + 1}(n) > \{n, n [1 [1 /_2 1 [1 /_2 1 [1 /_2 3] 2] 2] 2] 2\}$,
$f_{\theta(\zeta_{\Omega+1})}(n) = f_{\theta(\theta_1(2))}(n) > \{n, n [1 [1 /_2 1 /_2 2] 2] 2\}$ ($\theta_1$ is an ordinal collapsing function within θ),
$f_{\theta(\theta_1(3))}(n) > \{n, n [1 [1 /_2 1 /_2 1 /_2 2] 2] 2\}$,
$f_{\theta(\theta_1(\omega))}(n) > \{n, n [1 [1 [2 /_3 2] 2] 2] 2\}$ ($/_k$ is shorthand for $[1 /_{k+1} 2]$),
$f_{\theta(\theta_1(\theta(\theta_1(1)))) + 1}(n) > \{n, n [1 [1 [1 [1 [1 /_2 3] 2] 2 /_3 2] 2] 2] 2\}$ ($\theta_1(1) = \varepsilon_{\Omega+1}$),
$f_{\theta(\theta_1(\theta(\theta_1(\theta(\theta_1(1)))))) + 1}(n) > \{n, n [1 [1 [1 [1 [1 [1 [1 [1 /_2 3] 2] 2 /_3 2] 2] 2] 2 /_3 2] 2] 2] 2\}$,
$f_{\theta(\theta_1(\Omega))}(n) > \{n, n [1 [1 [1 / 2 /_3 2] 2] 2] 2\}$,
$f_{\theta(\theta_1(\Omega\char94\Omega))}(n) > \{n, n [1 [1 [1 / 1 / 2 /_3 2] 2] 2] 2\}$,
$f_{\theta(\theta_1(\Omega\char94\Omega\char94\Omega))}(n) > \{n, n [1 [1 [1 [1 / 2 /_2 2] 2 /_3 2] 2] 2] 2\}$,
$f_{\theta(\theta_1(\theta_1(1))) + 1}(n) > \{n, n [1 [1 [1 [1 /_2 3] 2 /_3 2] 2] 2] 2\}$,
$f_{\theta(\theta_1(\theta_1(\theta_1(1)))) + 1}(n) > \{n, n [1 [1 [1 [1 [1 [1 /_2 3] 2 /_3 2] 2] 2 /_3 2] 2] 2] 2\}$,
$f_{\theta(\Omega_2)}(n) > \{n, n [1 [1 [1 /_2 2 /_3 2] 2] 2] 2\}$,
$f_{\theta(\Omega_2\char94\Omega_2)}(n) > \{n, n [1 [1 [1 /_2 1 /_2 2 /_3 2] 2] 2] 2\}$,
$f_{\theta(\Omega_2\char94\Omega_2\char94\Omega_2)}(n) > \{n, n [1 [1 [1 [1 /_2 2 /_3 2] 2 /_3 2] 2] 2] 2\}$,
$f_{\theta(\varepsilon(\Omega_2+1)) + 1}(n) = f_{\theta(\theta_2(1)) + 1}(n) > \{n, n [1 [1 [1 /_3 3] 2] 2] 2\}$,
$f_{\theta(\zeta(\Omega_2+1))}(n) = f_{\theta(\theta_2(2))}(n) > \{n, n [1 [1 [1 /_3 1 /_3 2] 2] 2] 2\}$,
$f_{\theta(\theta_2(\omega))}(n) > \{n, n [1 [1 [1 [2 /_4 2] 2] 2] 2] 2\}$,
$f_{\theta(\theta_2(\Omega))}(n) > \{n, n [1 [1 [1 [1 / 2 /_4 2] 2] 2] 2] 2\}$,
$f_{\theta(\theta_2(\Omega_2))}(n) > \{n, n [1 [1 [1 [1 /_2 2 /_4 2] 2] 2] 2] 2\}$,
$f_{\theta(\theta_2(\theta_2(1))) + 1}(n) > \{n, n [1 [1 [1 [1 [1 /_3 3] 2 /_4 2] 2] 2] 2] 2\}$,
$f_{\theta(\theta_2(\theta_2(\theta_2(1)))) + 1}(n) > \{n, n [1 [1 [1 [1 [1 [1 [1 /_3 3] 2 /_4 2] 2] 2 /_4 2] 2] 2] 2] 2\}$,
$f_{\theta(\Omega_3)}(n) > \{n, n [1 [1 [1 [1 /_3 2 /_4 2] 2] 2] 2] 2\}$,
$f_{\theta(\varepsilon(\Omega_3+1)) + 1}(n) = f_{\theta(\theta_3(1)) + 1}(n) > \{n, n [1 [1 [1 [1 /_4 3] 2] 2] 2] 2\}$,
$f_{\theta(\zeta(\Omega_3+1))}(n) = f_{\theta(\theta_3(2))}(n) > \{n, n [1 [1 [1 [1 /_4 1 /_4 2] 2] 2] 2] 2\}$,
$f_{\theta(\theta_3(\omega))}(n) > \{n, n [1 [1 [1 [1 [2 /_5 2] 2] 2] 2] 2] 2\}$,
$f_{\theta(\Omega_4)}(n) > \{n, n [1 [1 [1 [1 [1 /_4 2 /_5 2] 2] 2] 2] 2] 2\}$,
$f_{\theta(\Omega_5)}(n) > \{n, n [1 [1 [1 [1 [1 [1 /_5 2 /_6 2] 2] 2] 2] 2] 2] 2\}$,
$f_{\theta(\Omega_6)}(n) > \{n, n [1 [1 [1 [1 [1 [1 [1 /_6 2 /_7 2] 2] 2] 2] 2] 2] 2] 2\}$.

The limit ordinal of Bird's Hierarchical Hyper-Nested Arrays is $\theta(\Omega_\omega)$.

The U(n) function at the end of Beyond Bird's Nested Arrays IV grows as rapidly as $f_{\theta(\Omega_\omega)}(n)$, which is also the growth rate of the Extended Kruskal Theorem (Kruskal's Tree Theorem extended to labelled trees, which have vertices labelled from 1 to n), the Graph Minor Theorem (or Subcubic Graph Numbers) and Buchholz Hydras (with ω labels removed).

Beyond Bird's Nested Arrays V introduces extensions to the Hierarchical Hyper-Nested Arrays created in the fourth Beyond Bird's Nested Arrays document, in which the single-value forward slash subscript itself becomes an array, a nested array, or even a nested subscript array. Also, these subscript arrays may be affixed to the inside of the square bracket separators, as long as each of these contains a ‘higher’ subscript array attached to a slash (anywhere within the separator). Examples include the following:

$f_{\theta(\Omega_\omega)}(n)$ > {n, n [1 [2 $/_{1,2}$ 2] 2] 2},
$f_{\theta(\theta_1(\Omega_\omega),\, 1)}(n)$ > {n, n [1 [2 $/_{1,2}$ 2] 3] 2},
$f_{\theta(\theta_2(\Omega_\omega),\, 1)}(n)$ > {n, n [1 [1 [2 $/_{1,2}$ 2 $_{2}$] 3] 2] 2},
$f_{\theta(\theta_3(\Omega_\omega),\, 1)}(n)$ > {n, n [1 [1 [1 [2 $/_{1,2}$ 2 $_{3}$] 3] 2] 2] 2}
(●$_k$ is used as shorthand for [2 $/_{1,2}$ 2 $_k$] at the beginning of the document),
$f_{\theta(\Omega_\omega,\, 1)}(n)$ > {n, n [1 [3 $/_{1,2}$ 2] 2] 2},
$f_{\theta(\Omega_\omega+1)}(n)$ > {n, n [1 [1 $/_{1,2}$ 3] 2] 2},
$f_{\theta(\Omega_\omega 2)}(n)$ > {n, n [1 [1 $/_{1,2}$ 1 $/_{1,2}$ 2] 2] 2},
$f_{\theta(\Omega_\omega{}^\wedge\omega)}(n)$ > {n, n [1 [1 [2 $/_{2,2}$ 2] 2] 2] 2},
$f_{\theta(\Omega_\omega{}^\wedge\Omega_\omega)}(n)$ > {n, n [1 [1 [1 $/_{1,2}$ 2 $/_{2,2}$ 2] 2] 2] 2},
$f_{\theta(\varepsilon(\Omega_\omega+1)) + 1}(n) = f_{\theta(\theta_\omega(1)) + 1}(n)$ > {n, n [1 [1 [1 $/_{2,2}$ 3] 2] 2] 2},
$f_{\theta(\zeta(\Omega_\omega+1))}(n) = f_{\theta(\theta_\omega(2))}(n)$ > {n, n [1 [1 [1 $/_{2,2}$ 1 $/_{2,2}$ 2] 2] 2] 2},
$f_{\theta(\theta_\omega(\omega))}(n)$ > {n, n [1 [1 [1 [2 $/_{3,2}$ 2] 2] 2] 2] 2},
$f_{\theta(\theta_\omega(\Omega))}(n)$ > {n, n [1 [1 [1 [1 / 2 $/_{3,2}$ 2] 2] 2] 2] 2},
$f_{\theta(\theta_\omega(\Omega_\omega))}(n)$ > {n, n [1 [1 [1 [1 $/_{1,2}$ 2 $/_{3,2}$ 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega+1})}(n)$ > {n, n [1 [1 [1 [1 $/_{2,2}$ 2 $/_{3,2}$ 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega+2})}(n)$ > {n, n [1 [1 [1 [1 [1 $/_{3,2}$ 2 $/_{4,2}$ 2] 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega+3})}(n)$ > {n, n [1 [1 [1 [1 [1 [1 $/_{4,2}$ 2 $/_{5,2}$ 2] 2] 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega 2})}(n)$ > {n, n [1 [1 [2 $/_{1,3}$ 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega 2}+1)}(n)$ > {n, n [1 [1 [1 $/_{1,3}$ 3] 2] 2] 2},
$f_{\theta(\Omega_{\omega 2}2)}(n)$ > {n, n [1 [1 [1 $/_{1,3}$ 1 $/_{1,3}$ 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega 2}{}^\wedge\Omega_{\omega 2})}(n)$ > {n, n [1 [1 [1 [1 $/_{1,3}$ 2 $/_{2,3}$ 2] 2] 2] 2] 2},
$f_{\theta(\varepsilon(\Omega_{\omega 2}+1)) + 1}(n) = f_{\theta(\theta_{\omega 2}(1)) + 1}(n)$ > {n, n [1 [1 [1 [1 $/_{2,3}$ 3] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega 2+1})}(n)$ > {n, n [1 [1 [1 [1 [1 $/_{2,3}$ 2 $/_{3,3}$ 2] 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega 2+2})}(n)$ > {n, n [1 [1 [1 [1 [1 [1 $/_{3,3}$ 2 $/_{4,3}$ 2] 2] 2] 2] 2] 2] 2}
(//$_k$ and ///$_k$ are used as shorthand for $/_{k,2}$ and $/_{k,3}$ respectively on pages 5-21),
$f_{\theta(\Omega_{\omega 3})}(n)$ > {n, n [1 [1 [1 [2 $/_{1,4}$ 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega 4})}(n)$ > {n, n [1 [1 [1 [1 [2 $/_{1,5}$ 2] 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 2})}(n)$ > {n, n [1 [2 $/_{1,1,2}$ 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 2}+1)}(n)$ > {n, n [1 [1 $/_{1,1,2}$ 3] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 2}2)}(n)$ > {n, n [1 [1 $/_{1,1,2}$ 1 $/_{1,1,2}$ 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 2}{}^\wedge\Omega_{\omega^\wedge 2})}(n)$ > {n, n [1 [1 [1 $/_{1,1,2}$ 2 $/_{2,1,2}$ 2] 2] 2] 2},
$f_{\theta(\varepsilon(\Omega_{\omega^\wedge 2}+1)) + 1}(n) = f_{\theta(\theta_{\omega^\wedge 2}(1)) + 1}(n)$ > {n, n [1 [1 [1 $/_{2,1,2}$ 3] 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 2+1})}(n)$ > {n, n [1 [1 [1 [1 $/_{2,1,2}$ 2 $/_{3,1,2}$ 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 2+2})}(n)$ > {n, n [1 [1 [1 [1 [1 $/_{3,1,2}$ 2 $/_{4,1,2}$ 2] 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 2+\omega})}(n)$ > {n, n [1 [1 [2 $/_{1,2,2}$ 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 2+\omega 2})}(n)$ > {n, n [1 [1 [1 [2 $/_{1,3,2}$ 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{(\omega^\wedge 2)2})}(n)$ > {n, n [1 [1 [2 $/_{1,1,3}$ 2] 2] 2] 2},
$f_{\theta(\Omega_{(\omega^\wedge 2)3})}(n)$ > {n, n [1 [1 [1 [2 $/_{1,1,4}$ 2] 2] 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 3})}(n)$ > {n, n [1 [2 $/_{1,1,1,2}$ 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge 4})}(n)$ > {n, n [1 [2 $/_{1,1,1,1,2}$ 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge\omega})}(n)$ > {n, n [1 [2 $/_{1\ [2]\ 2}$ 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge\omega^\wedge 2})}(n)$ > {n, n [1 [2 $/_{1\ [3]\ 2}$ 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge\omega^\wedge\omega})}(n)$ > {n, n [1 [2 $/_{1\ [1,2]\ 2}$ 2] 2] 2},
$f_{\theta(\Omega_{\omega^\wedge\omega^\wedge\omega^\wedge\omega})}(n)$ > {n, n [1 [2 $/_{1\ [1\ [2]\ 2]\ 2}$ 2] 2] 2},
$f_{\theta(\Omega_{\varepsilon_0}) + 1}(n)$ > {n, n [1 [2 $/_{1\ [1\ /\ 2]\ 2}$ 2] 2] 2},
$f_{\theta(\Omega_{\Gamma_0})}(n)$ > {n, n [1 [2 $/_{1\ [1\ [1\ /\ 2\ /_2\ 2]\ 2]\ 2}$ 2] 2] 2},

$f_{\theta(\Omega_{\theta(\varepsilon_{\Omega+1})}) + 1}(n) > \{n, n\ [1\ [2\ /_{1\ [1\ [1\ /_2\ 3]\ 2]\ 2}\ 2]\ 2]\ 2\}$,
$f_{\theta(\Omega_{\theta(\Omega_2)})}(n) > \{n, n\ [1\ [2\ /_{1\ [1\ [1\ [1\ /_2\ 2\ /_3\ 2]\ 2]\ 2]\ 2}\ 2]\ 2]\ 2\}$,
$f_{\theta(\Omega_{\theta(\Omega_3)})}(n) > \{n, n\ [1\ [2\ /_{1\ [1\ [1\ [1\ [1\ /_3\ 2\ /_4\ 2]\ 2]\ 2]\ 2]\ 2}\ 2]\ 2]\ 2\}$.

Taking $S_1$ = '1 [1 [2 $/_{1,2}$ 2] 2] 2' and $S_{n+1}$ = '1 [1 [2 $/_{S_n}$ 2] 2] 2',
$f_{\theta(\Omega_{\theta(\Omega_\omega)})}(n) > \{n, n\ [1\ [2\ /_{S_1}\ 2]\ 2]\ 2\}$,
$f_{\theta(\Omega_{\theta(\Omega_{\theta(\Omega_\omega)})})}(n) > \{n, n\ [1\ [2\ /_{S_2}\ 2]\ 2]\ 2\}$,
$f_{\theta(\Omega_{\theta(\Omega_{\theta(\Omega_{\theta(\Omega_\omega)})})})}(n) > \{n, n\ [1\ [2\ /_{S_3}\ 2]\ 2]\ 2\}$,
which means that the limit ordinal of the $S_n$ sequence – and Bird's Nested Hierarchical Hyper-Nested Array Notation – is $\theta(\Omega_\Omega)$.

The S(n) function at the end of Beyond Bird's Nested Arrays V grows as rapidly as $f_{\theta(\Omega_\Omega)}(n)$.

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